Testing for Structural Change and Nonstationarity

by

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Abstract

KENNETH N. HIGHTOWER: Testing for Structural Change and Nonstationarity
(Under the direction of William R. Parke)

Stationarity and structural stability are two of the most important issues in time series econometrics. This dissertation is a study of the relationship between testing for stationarity and testing for structural change. Specifically, it is shown that a commonly used test for stationarity against the alternative of a unit root proposed by Kwiatkowski, Phillips, Schmidt, and Shin (1992) (KPSS) is algebraically equivalent to a member of a class of tests against structural change at an unknown change-point suggested by Andrews and Ploberger (1994). We extend this class by proposing two new tests for structural change. The maximum value of the KPSS test statistic is derived, and shown to be a non-stochastic cosine function. This result is used to study the asymptotic local power of the test for various alternatives, including structural breaks, unit roots, and fractional integration.

A new class of tests is proposed to distinguish between structural change and unit roots. It involves a two-stage testing methodology that looks at the properties of sub-samples of the data in a second-stage test. The basic idea is that the subsample results should show a localized rejection in the case of a single structural break and widespread rejection in the case of a unit root.

Finally, we examine the empirical properties of Treasury securities. We find evidence of persistence in returns, yields and term-premia, but no evidence for persistence in excess returns. We use the implications above to explore whether the persistence in U.S. debt market returns and yields can be explained through structural change. To this end we look at several methods of splitting the series into sub-samples and testing for persistence within sub-samples. The idea, similar to that pursued by Lobato and Savin (1998) for stock returns and squared returns, is that if the full-sample results are spuriously induced by structural instability, there should not be any evidence of long memory in the sub-samples. We find that the evidence of long memory remains even after accounting for underlying structural changes.

What is a man,
If his chief good and market of his time
Be but to sleep and feed? a beast, no more.
Sure, he that made us with such large discourse,
Looking before and after, gave us not
That capability and god-like reason
To fust in us unused. Now, whether it be
Bestial oblivion, or some craven scruple
Of thinking too precisely on the event,
A thought which, quarter'd, hath but one part wisdom
And ever three parts coward, I do not know
Why yet I live to say 'This thing's to do;'
Sith I have cause and will and strength and means
To do't.

Hamlet, Act IV Scene iv

To my loving wife Jeanine:
you are my cause and will and strength and means.

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Chapter 1

Introduction

Stable relationships among variables are the foundation for nearly all economic theories. Stability is important because without it conclusions drawn from economic analyses can be misleading or worthless. The most common assumptions in economic models are ones that restrict the structure to be stable over time and/or across observations. Known violations of these assumptions can be accounted for by modeling any changes in structure. Therefore, it is not surprising that nearly all serious empirical analyses begin by either explicitly or implicitly applying tests for stability.

For these reasons, stationarity and structural stability are two of the most important issues in time series econometrics. Over the past decade, two large literatures have developed around testing for each of these phenomena. Unit root testing was pioneered by Dickey and Fuller (Fuller, 1976) and has been extended by many others who have devised test statistics to divine whether or not a series is I(1), or in the common terminology, has a unit root. A similar idea was proposed by Kwiatkowski, Phillips, Schmidt, and Shin (1992), hereafter KPSS, to test for I(0), or stationary, data. The properties of these statistics have been exhaustively explored by many researchers.

The origins of structural stability tests in econometrics can be traced back to Chow (1960). These types of tests traditionally took the hypothesized break point as known a priori. Recently, Andrews (1993) and Andrews and Ploberger (1994) developed a class of tests that take the break point as unknown, thus explicitly incorporating the "eye-balling" of all possible break points that has been a common criticism (see Christiano, 1992) of Chow type structural break tests.

In Chapter 3 we investigate the relationship between a commonly used test of stationarity and tests of structural change. We show that the commonly used Kwiatkowski, Phillips, Schmidt, and Shin (1992) (KPSS) test for stationarity vs. a unit root emerges as a special case from a framework that Andrews and Ploberger (1994) propose to test for a structural change when the change-point is unknown. We extend the Andrews and Ploberger framework to include two new test for structural change, which we denote $ExpS$, and $SupS$. For a series $y_t = r_t + u_t$ where r_t is long memory and u_t is stationary, the tests are all based on the partial sums, $S_t = \sum_{i=1}^{t}(y_i - \bar{y})$, of the residuals from a regression of y_t on a constant. Our new tests are based on functionals that weight each test of structural change at point π, $LM(\pi)$, by the weighting function $J(\pi) = \pi(1-\pi)$. In this context, the KPSS test is shown to occupy a niche we denote, $AvgS$. The new statistics are given below

$$AvgS = \frac{T^{-2}\sum_{t=1}^{T} S_t^2}{s^2(\ell)}$$

$$ExpS = \ln\left(\frac{6}{T}\sum_{t=1}^{T-1}\exp\left(\frac{\sum_{t=1}^{T} S_t^2}{2Ts^2(\ell)}\right)\frac{t}{T}\left(1-\frac{t}{T}\right)\right)$$

$$SupS = \sup_{t\in[1,T]}\frac{S_t^2}{Ts^2(\ell)}$$

where $s^2(\ell)$ is a kernel-based estimate of the long-run variance of u_t. These tests compare to the Andrews (1993) and Andrews and Ploberger (1994) tests $AvgLM(\pi_0)$, $ExpLM(\pi_0)$, and $SupLM(\pi_0)$ where π_0 is a trimming parameter and the $LM(\pi)$ statistics are given equal weighting (i.e. $J(\pi) = 1$).

Our new tests have some power advantages when a break occurs in the middle of the sample as well as for the case of a one-time structural break that is uniformly distributed within the sample. We also investigate the properties of both types of tests under the alternatives of fractional integration and a unit root. An interesting result is that there is very little difference in the performance of the tests across alternatives. In fact, a test we denote $AHBT(1/2)$ for the "*ad hoc*" break test for a structural break in the middle of the sample has close to the power of the $AvgS$/KPSS test for a unit root alternative.

The similarity between testing outcomes has an important implication for empirical researchers. A rejection for a stationarity test does not imply a unit root. Likewise, a rejection of a test for structural change does not imply a break. This is not merely a loss of power. The two types of test are fundamentally testing for the same thing. Instead, a rejection of either test implies that the series is not short memory.

Chapter 4 looks at the kind of processes that yield large values of the KPSS statistic. We find that the KPSS test normalized by the sample size, T, has a maximum value and we derive the process that produces this value for both the demeaned, $\hat{\eta}_\mu$, and detrended, $\hat{\eta}_\tau$, versions of the test. We show that although the test is designed with the unit root alternative in mind, the maximum value is obtained for a non-stochastic cosine function. Specifically, we show that $\max T^{-1}\hat{\eta}_\mu = \pi^{-2}$ for the process $y_t = cos(\pi t/T)$, while $\max T^{-1}\hat{\eta}_\tau = (2\pi)^{-2}$ for the process $y_t = cos(2\pi t/T)$. These cosine functions provide benchmarks for other possible alternative hypotheses, and they give a clear picture of the types of realizations that actually trigger large values for $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$. They show that $\hat{\eta}_\mu$ is most sensitive to a large change in the value of y_t between the beginning and the end of the sample and that $\hat{\eta}_\tau$ is most sensitive to data with a complete cycle in the data period.

These extreme value results establish a framework for studying the power of the KPSS tests for alternatives that include unit roots and structural breaks. We consider an alternative hypothesis based on the process

$$z_t = x_t + \gamma_T y_t$$

composed of the sum of a short memory process x_t and a long memory process y_t. The scalar γ_T sets the distance between the null and the alternative for a sample of size T. We show that the power for a small γ_T is closely related to the extreme value results for the process y_t alone.

Chapter 5 investigates the central problem raised by the results from Chapter 3. Namely, how can we distinguish between structural change and unit roots if the commonly used tests yield nearly indistinguishable results for the two alternatives? We propose a two-stage testing methodology that looks at the properties of sub-samples of the data in a second-stage test. That is, given a Stage I rejection of a stationarity/structural change test λ, we propose calculating λ for sub-samples of the data. If $\lambda(m, k)$ is a statistic with a null hypothesis of short memory applied to a subsample starting in period m and of length k. Our proposed subsample statistic for testing the hypothesis Stage II null hypothesis of a unit root process is given by

$$\Lambda_t(q) = \frac{\operatorname{avg}_{m \in M} \Lambda_t(m, k)}{\sup_{m \in M} \Lambda_t(m, k)}$$

where $q = k/T$ is constant and $M = [1, T - k]$. The basic idea is that the subsample results should show a localized rejection in the case of a single structural break and widespread rejection in the case of a unit root.

We derive the distribution of $\Lambda_t(q)$ under a unit root for several choices of short memory statistics $\Lambda_t(m, k)$, including several examined in Chapter 3. We show that $\Lambda_t(q)$ is consistent against a one-time structural change where the change-point is uniformly distributed.

Our Stage II test performs reasonably well, particularly when the probability of a Stage I rejection is high. We find that the performance of the test can be markedly improved by using a hybrid version that uses different Λ_t in the Stage I and Stage II tests. The best performing test is one that combines the $supLM$ test in Stage I with the $AHBT(1/2)$ in Stage II. This is because, while the $AHBT(1/2)$ has the best properties for Stage II, it suffers from a loss of power in the Stage I test.

Chapter 6 uses the implications from Chapter 3 to explore whether the persistence in U.S. debt market returns and yields can be explained through structural change. Unlike equity returns, we find strong evidence for long memory in U.S. Treasury securities. We also document this persistence for yields and term-premia but find no evidence of long memory in excess returns. To test if structural change is causing this, we split the series in a number of ways to test the sub-samples for long memory. First, we use a split in October 1979 when the FED changed its operating procedure. We follow by splitting using a sequential break-point estimator of Bai and Perron (2001) on a monetary variable, M2. The resulting break dates are then applied to the return and yield series. Evidence of long memory in the full-sample returns but not in the full-sample excess returns motivates a Fischer type story where breaks in inflation effect returns but not excess returns. Again, we use the sequential break-point estimator to split the inflation series and apply the results to returns and yields. Finally, we estimate the conditional state probabilities from a Hamilton (1990) two-state Markov-switching model. We use these probabilities to split the samples into contiguous regimes.

Overall, we find very little support for the structural break hypothesis. Persistence remains in nearly all of the sub-samples across the different splits, implying that long memory models should be seriously considered when modeling debt market returns, yields, and term-premia. This is finding has important implications for term structure models, bond pricing, and fixed income derivative models. The Markov-switching models show some promise, although in many cases the states change too frequently to produce reliable sub-samples in which to test. This, of course, may very well be seen as evidence to support the view that there is little observational difference between long memory and Markov-switching models.

Chapter 2

Literature Review

This chapter briefly reviews long memory, stationarity, and structural change to provide background and context for the issues explored in this dissertation.

2.1 Long Memory

Normally, only integer powers of d are considered in ARIMA(p, d, q) models, but there is no mathematical or statistical requirement that d take on only integer values (e.g., $d=1$ yields a first-difference model). In a fractionally-differenced model, d can take on non-integer values and the resulting time series can exhibit some particularly interesting dependencies. Granger and Joyeux (1980) and Hosking (1981) show that extending the lag operator to non-integer powers of d results in a well-defined time series that is fractionally integrated of order d. The differencing operator may be written

$$(1 - L)^d = \sum_{k=0}^{\infty} (-1)^k \binom{d}{k} L^k \tag{2.1}$$

leading to the following representation of a time series where $p = q = 0$:

$$(1 - L)^d y_t = \sum_{k=0}^{\infty} \frac{\Gamma(k - d)}{\Gamma(-d)\Gamma(k + 1)} y_{t-k} \tag{2.2}$$

Here, Γ is the usual gamma function.

In his excellent survey paper, Baillie (1996) reviews a number of different long-memory models. One simple model is an ARFIMA (0, d, 0) process given by

$$(1 - L)^d (y_t - \mu) = \epsilon_t \tag{2.3}$$

This model is studied in Granger (1980), Granger and Joyeux (1980), and Hosking (1981). Their work shows than when $d < .5$, the series has finite variance, but for $d = .5$, the series has infinite variance. The time series is stationary and invertible when $-.5 < d < .5$. For $d = .5$, standard Box-Jenkins techniques will indicate that differencing is required and provided that $d < 1$, differencing will produce a series whose spectrum is zero at zero frequency. This heavily-used model is a special case of an ARFIMA (p, d, q) process given by

$$\Phi(L)(1 - L)^d (y_t - \mu) = \Theta(L)\epsilon_t \tag{2.4}$$

where $p = q = 0$.

Fractionally-differenced time-series models have very interesting long-run forecasting properties. A fractional white noise series $y_t \sim I(d)$ may be represented as an MA(∞) process where the moving average coefficients decline slowly following the form $b_j \sim A_j^{d-1}$ where A is a constant. A stationary ARMA(p, q) with infinite p and q will have coefficients that decline at least exponentially: $b_j \sim A\theta^j$. One important implication of these stark differences in coefficient decay rates is that a fractionally-differenced model may provide better long-run forecasts from a very simple model compared to ARMA(p, q) models where p and q are large.

2.2 Stationarity

A method for testing for stationarity in economic time series proposed by Kwiatkowski, Phillips, Schmidt, and Shin (1992), hereafter KPSS, has become a very popular analytic tool among applied econometricians. They tackle the unit root testing problem from the opposite direction of the ubiquitous tests of Dickey and Fuller. Whereas the Dickey-Fuller test and its variants starts from the null hypothesis of a unit root, the KPSS test has a null of stationarity. The test is an LM test that a series based on the sum of a deterministic trend, random walk, and stationary error. More formally let y_t, $t = 1, \ldots, T$ be the observed series to be tested for stationarity. Assume that y_t can be decomposed as:

$$y_t = \xi t + r_t + \epsilon_t \tag{2.5}$$

where

$$r_t = r_t + u_t$$

and ϵ_t and u_t are iid $(0, \sigma_\epsilon^2)$, iid $(0, \sigma_u^2)$ respectfully. Under the null hypothesis of stationarity, $\sigma_u^2 = 0$. The computation of the test is quite simple. Let $\hat{\epsilon}_t$, $t = 1, \ldots, T$ be the residuals from the regression of y on an intercept and/or a time trend. Let $s^2(\ell)$ be an estimate of the long run variance. If we define the partial sum process of the residuals:

$$S_t = \sum_{i=1}^{t} \hat{\epsilon}_i, \qquad t = 1, \ldots, T$$

then the KPSS test statistics are given by:

$$\hat{\eta} = T^{-2} \sum S_t^2 / s^2(\ell)$$

for tests based on the residuals from a level $(\hat{\eta}_u)$ or time trend $(\hat{\eta}_\tau)$ regression.

The distribution of the test statistic converges to a functional of a Brownian Bridge. The estimator of the long run variance is of the type suggested by Newey and West (1987). They tabulate upper tail critical values for the two distributions and apply the tests to the Nelson and Plosser (1982) data set.

Several other tests along the same vein were suggested previously by Nyblom and Mäkeläinen (1983), Nyblom (1986, 1989), and Nabeya and Tanaka (1988). These were, however, generally derived under more restrictive assumptions on the error distribution that are unlikely to hold in the interesting cases in economic data. Specifically, KPSS allow for fairly general dependence in the error structure under the null hypothesis of stationarity. They find that under regularity conditions like those of Phillips and Perron (1988), a heteroscedastic autocorrelation consistent (HAC) estimator of the long run variance proposed by Andrews (1991) accounts for any short term dependence.

Leybourne and McCabe (1994) take another approach in accounting for short term dynamics in the error structure. They derive a parametric stationarity test using the partial sums of residuals from an ARIMA$(p,1,1)$ model. This test is similar in spirit and distribution under the null to the KPSS test of stationarity. In fact, it is convenient to think of the Leybourne-McCabe test relating to the KPSS test as the ADF test for a unit root relates to the non-parametric Phillips and Perron (1988) unit root test. The stationarity tests have similar size and power, however the Leybourne-McCabe test is not as sensitive to the choice of $p > p^*$ as the KPSS test is to the choice of the lag truncation parameter ℓ in the estimate of the long-run variance.

Lee and Schmidt (1996) show that the KPSS test is consistent against stationary long memory alternatives, such as $I(d)$ processes for $d \in (-0.5, 0.5)$, $d \neq 0$. It can therefore be used to distinguish short memory and long memory stationary processes. The power of the KPSS test in finite samples is found to be comparable to that of modified rescaled range test (Lo, 1991). Their results show that a rather large sample size, such as $T = 1000$, will be necessary to distinguish reliably between a long memory process and a short memory process with comparable short-term autocorrelation.

Lee and Amsler (1997) derive the asymptotic distribution of the KPSS (1992) statistic under nonstationary long memory $(0.5 < d < 1)$. They find that its order in probability is the same under nonstationary long memory as under a unit root. It cannot, therefore, distinguish consistently between the two cases.

Lee, Huang, and Shin (1997) examine the effect of a structural break on stationarity tests. Previous work has shown that stationarity tests suffer from size distortion problems if a structural break exists but is ignored. This problem parallels the power loss problem of unit root tests ignoring an existing break. They find that the distributions of stationarity tests are asymptotically invariant to the exclusion of the existing break under the alternative hypothesis of a unit root.

2.3 Structural Change

Two general tests for structural change that are easy to compute and frequently used in practice are the CUSUM and CUSUM of squares (CUMSUMSQ) test statistics suggested by Brown, Durbin, and Evans (1975). The tests are based on recursive residuals and are quite general in that they do not require specification of the type or timing of the structural change.

The t^{th} recursive residual, e_t, can be thought of as the *ex post* forecast error for y_t when a regression is estimated using only the first t-1 observations. If w_t is the recursive residual scaled by the forecast variance, then $w_t \sim N(0, \sigma^2)$ and $E(w_r, w_s)$=0 for all $r \neq s$. The CUSUM test is based on the cumulated sum of these scaled residuals and the test is performed by plotting this cumulated sum against time and observing whether or not it strays outside of confidence bands for the appropriate significance level. The CUSUMSQ test is formed in a similar fashion.

Ploberger and Krämer (1992) extend the distribution theory of CUSUM-type tests from recursive residuals to OLS residuals and show that the distributions go to functionals of a Brownian bridge. In this sense, CUSUM-type tests are closely related to the KPSS test for stationarity.

One drawback of CUSUM-type tests is that, as general tests of structural change, they are not as powerful as Chow-type tests that specify specific break points. However, Chow tests lose much of this advantage if the break point is unknown.

Andrews (1993) addresses this issue by formulating a general test for parameter instability and structural change with unknown change point in nonlinear parametric models. His tests are LM, LR, and Wald tests based on the generalized method of moments (GMM) estimators. The tests take the form

$$\sup_{\pi \in \Pi} LM_T(\pi)$$

where $LM_T(\pi)$ is a test for structural change at time $\pi \in \Pi$, where Π is a subset of $(0,1)$. This statistic formally models one of the common criticisms of tests for structural change, namely that the best candidate for a break is often picked *a priori*.

The distribution is given by

$$\max_{\pi \in \Pi}(B_p(\pi) - \pi B_p(1))'(B_p(\pi) - \pi B_p(1))/[\pi(1-\pi)]$$

where B_p is a p-vector of Brownian motion, p being the number of parameters that change under the alternative hypothesis. The distribution depends on Π, the set of possible change points. This set must be bounded away from the endpoints. He tabulates critical values for various values of p and Π.

Andrews and Ploberger (1994) consider optimal tests for parameter vector constancy when the likelihood function depends on an additional parameter under the alternative. A classic example of this problem is testing for a one time structural change where under the alternative the distribution depends on the breakpoint π. The optimal test statistic is given by

$$ExpLM_T = (1+c)^{-p/2} \int \exp\left(\frac{1}{2}\frac{c}{1+c} LM_T(\pi)\right) dJ(\pi)$$

where $LM_T(\pi)$ is the standard test for a break at time π and $J(\pi)$ is a weight function over the set of possible breakpoints which may be given the interpretation of a Bayesian prior.

The statistic depends critically on the parameter c which for larger values of c gives more weight to alternatives where the break size, β, is large. Taking the normalized limit as $c \to 0$

yields the "average LM" statistic which is designed for alternatives close to the null.

$$\lim_{c\to 0} 2(ExpLM_{Tc} - 1)/c = \int LM_T(\pi)dJ(\pi)$$

For different models, a variant of this statistic has been considered by, among others, Nyblom (1989).

At the other extreme, the normalized statistic as $c \to \infty$ is given by

$$\lim_{c\to\infty} \log\left((1+c)^{p/2} ExpLM_{Tc}\right) = \log \int \exp\left(\frac{1}{2}LM_T(\pi)\right) dJ(\pi)$$

Note further, that if $c/(1+c)$ is replaced by $r > 0$, the normalized statistic as $r \to \infty$ is the $supLM$ statistic considered by Andrews (1993)

$$\lim_{r\to\infty} (\log ExpLM^r_{Tc})/r = \sup_{\pi\in\Pi^*} LM_T(\pi)$$

The latter test is not an optimal test of the sort considered. The $supLM$ test directs power against extreme alternatives.

Andrews, Lee, and Ploberger (1996) derive finite-sample optimal tests that generalize Andrews and Ploberger (1994) for one or more change points at unknown times in a multiple normal linear regression model with fixed regressors. More generally, these tests may be used to test the null of parameter constancy against a very broad range of alternatives parameter changes of a less specific nature. For instance martingale parameter changes similar to that studied by Nyblom (1989).

They suggest the use of the $expF$ version of the test ($c = \infty$). The statistics are constructed from

$$F(\pi) = \frac{((T-s)/(Tmv))\, LM(\pi)}{(1 - LM(\pi)/T)}$$

where m is the number of change points and v is the number of parameters that change under the alternative hypothesis. The optimal statistics are then:

$$ExpF_c = (1+c)^{-mv/2} \sum_{\pi\in\Pi} \exp\left(\frac{mv}{2}\frac{c}{1+c}F(\pi)\right) dJ(\pi)$$

Two useful normalized limiting statistics are

$$avgF = \lim_{c\to 0} 2(ExpF_c - 1)/(cmv) = \sum_{\pi\in\Pi} F(\pi)J(\pi)$$

and

$$expF_{\infty} = \lim_{c \to \infty} \log \left((1+c)^{m\iota/2} ExpF_c \right) = \log \sum_{\tau \in \Pi} \exp \left(\frac{m\iota}{2} F(\tau) \right) J(\tau)$$

They compare power for many statistics across a range of alternatives. Two of their comparison tests are the $NybLM$ test of Nyblom (1989) and the so called midpoint F-test, $F(0.5)$. Their findings include the observation that the $F(0.5)$ test does pretty well against martingale alternatives and that $NybLM$ is more powerful against breaks in the middle of the sample. The latter is not surprising in light of the interpretation of the $NybLM$ test as a version of $avgLM$ with $J(\pi)$ equal to a particular nonuniform function that gives more weight to breaks in the middle of the sample.

Diebold and Chen (1996) compare two approximations to the finite sample distributions of test statistics for structural change under the null hypothesis of stability, one based on asymptotics and the other based on the bootstrap. The asymptotic tests are of the supremum, average, and exponential type suggested by Andrews (1993) and Andrews and Ploberger (1994). The bootstrap distribution is in the spirit of Christiano (1992). They look at Gaussian mean zero AR(1) processes where a break implies that the AR(1) coefficient changes in the sample. They conclude that caution should be exercised when using the asymptotic test procedures because there are significant deviations between the nominal and empirical test size. The bootstrap, however, appears to do well even in small samples with high serial correlation. It should be noted that they are looking at the properties when there are dynamic regressors whereas Andrews focuses on deterministic regressors. Their findings for the Asymptotic SupLM test is that it tends to under reject drastically but is not as sensitive to serial correlation as are the AsySupLR or AsySupW statistics.

Nunes, Kuan, and Newbold (1995) analyze a QMLE estimator of the break date. They show consistency provided the data generating process is not integrated. They find that when the series is generated by an I(1) process with drift, analysis can spuriously lead to a break near the middle of the sample.

Leybourne and Newbold (2000) look at tests of the unit root null when the true data generating process is I(1) with a break. They find that ADF type statistics over-reject the null if there is a break in the early part of the sample. Thus, structural breaks can hide evidence of a unit root as well as lead to the spurious unit root conclusions seen elsewhere. They find that this phenomena is not present for a test based on a symmetric weighted estimator. The properties of the weighted test are much better for the mean change than for the break in trend case where a test size near zero shows the difficulty in distinguishing between structural change and unit roots.

Chapter 3

Tests for Structural Change and Nonstationarity

We establish a unifying algebraic framework for testing for long memory. Our framework encompasses structural change, fractional integration, and unit roots. It includes several commonly used tests for structural change and unit roots as special cases. In particular, we demonstrate that the KPSS test (Kwiatkowski, Phillips, Schmidt, and Shin, 1992) based on a unit root alternative hypothesis is a special case of a test Andrews and Ploberger (1994) proposed for structural change. We also introduce two new tests for long memory that have desirable properties under certain alternatives. We show that the empirical differences among these tests are rather small and that the choice of one over another depends upon the particulars of the alternative under consideration.

3.1 The Nature of the Problem

Although structural changes and unit roots are both explanations for permanent changes in economic data, the two concepts have generally been treated completely separately. We show that the Kwiatkowski, Phillips, Schmidt, and Shin (1992) test for a unit root is an algebraic special case of the Andrews and Ploberger (1994) test for a structural break. The two papers, nonetheless, do not reference one another and do not share a single common reference.

In this chapter, we provide a unifying algebraic framework for tests for structural change and tests for nonstationarity, illustrating the fundamental similarities linking the two disparate strands of the literature. We also show that the practical implication of these similarities is that the differences among the tests are minimal for a range of alternatives, including structural change, unit roots, and fractional integration.[1] Structural change and other forms of nonstationarity are not, however, identical, and we will introduce a new test in Chapter 5 that effectively discriminates between the two forms of long memory.

The memory or dependence of a zero mean time series z_t can be expressed by the properties of its partial sum $S_t = \sum_{j=1}^{t} z_j$. The process z_t is said to be *short memory* if $\sigma^2 = \lim_{T\to\infty} E(T^{-1}S_T^2)$ exists and is nonzero, and

$$\frac{1}{\sigma T^{1/2}} S_{[rT]} \Rightarrow B(r), \qquad \forall r \in [0, 1] \tag{3.1}$$

where $[rT]$ is the integer part of rT, $\Rightarrow$ denotes convergence in distribution, and $B(r)$ is standard Brownian motion. Baillie (1996) shows that these conditions allow for departures from covariance stationarity, but require the existence of absolute moments of order β, for some $\beta > 2$.

In Section 3.2, we propose that the most direct approach to testing for long memory is to estimate the long-run variance, σ^2, that is at the heart of the definition. We propose the test

$$\Lambda = \frac{s^2(T)}{s^2(\ell)}, \tag{3.2}$$

where $s^2(\cdot)$ is the familiar kernel-based estimate of σ^2 using the Bartlett weights. We demonstrate that this test, properly normalized, is algebraically identical to the KPSS test for nonstationarity. We then demonstrate that both tests are an algebraic special case of the Andrews and Ploberger test for structural change at an unknown change-point.

Historically, of course, the development of tests has concentrated on direct comparisons of versions of a null hypothesis emphasizing stationarity with specific nonstationary alternatives. Consider processes of the form

$$y_t = z_t + r_t. \tag{3.3}$$

Kwiatkowski et al. (1992), for example, derive their form of Λ in the context of the case where z_t is stationary and r_t is a unit root process. While the stated null hypothesis for this test is a stationary process, Lee and Schmidt (1996) show that the natural null and alternative

[1] This finding is not entirely new. Hendry and Neale (1991) point out that a structural break is very likely to be diagnosed as a unit root if the test applied is a unit root test.

hypotheses for the KPSS test are short memory and long memory. This conclusion follows from their finding that the test is consistent for fractionally integrated processes $I(d)$ with $0 < d < 1/2$. Such a process is stationary, but not short memory, and demonstrating that it is rejected with probability one in large samples clearly establishes that the null hypothesis for the KPSS test is short memory, not stationarity.

The structural break literature arrives at the same test from a different origin. Andrews (1993) considers the case, in (3.3) above, where ε_t is stationary and u_t is a structural break process. Let $LM_T(\pi)$ be the statistic for the Lagrange multiplier test for the alternative hypothesis that a structural break occurs at a known observation $[\pi T]$ in a sample of size T. If the breakpoint is unknown, Andrews (1993) proposes a statistic of the form

$$\sup_{\pi \in \Pi} LM_T(\pi) \tag{3.4}$$

where the set $\Pi = [\pi_0, 1 - \pi_0]$ is bounded away from the endpoints by the choice of a trimming parameter π_0. The trimming parameter is necessary because, as Andrews (1993) shows, when $\Pi = [0, 1]$ the test statistic (3.4) does not converge in distribution.

We propose an alternative test for structural change that avoids the need to choose π_0. Our statistic is of the form

$$\sup_{\pi \in [0,1]} \pi(1 - \pi) LM_T(\pi), \tag{3.5}$$

where the supremum is over the entire [0,1] range. The level-α critical value for the case of a univariate structural change is $-ln(\alpha/2)/2$, which is surprisingly elegant given the complexity of the distributions in this literature.

Andrews and Ploberger (1994) introduce a weighted average of LM statistics as an alternative to the supremum. Their statistic is given by

$$\int_{\pi \in \Pi} LM_T(\pi) dJ(\pi), \tag{3.6}$$

where $J(\pi)$ is some measure defined over Π. For a structural break with an unknown breakpoint they recommend the uniform weighting function $J(\pi) = 1$ on intuitive grounds. The obvious extension to our new statistic (3.5) is the weighting function $J(\pi) = \pi(1 - \pi)$. This leads us to propose the statistic

$$\int_{\pi \in [0,1]} \pi(1 - \pi) LM_T(\pi) d\pi. \tag{3.7}$$

which equals λ, given in (3.2) above.

This statistic is, in fact, a well-known test for unit roots. We show that, for a univariate time series process, (3.7) is algebraically equal to the KPSS statistic

$$\widehat{\eta}_\mu = \frac{T^{-2}\sum_{t=1}^{T} S_t^2}{s^2(\ell)}$$

where $S_t = \sum_{i=1}^{t}(y_i - \bar{y})$ and $s^2(\ell)$ is a kernel-based estimate of the long-run variance.

Showing that the KPSS test for unit roots is a particular case of the Andrews-Ploberger test for a structural breaks has important implications. It reinforces the point that the differences between tests for structural breaks and tests for unit roots are more a matter of approach than of practical conclusions. The practical differences between a constant weighting function over the interval $[\pi_0, 1-\pi_0]$, as suggested by Andrews (1993) and Andrews and Ploberger (1994), and the weights $J(\pi) = \pi(1-\pi)$ over the interval $[0, 1]$, used in the KPSS statistic and in our new statistic (3.5), are relatively small over the middle portion of the range between 0 and 1. We show in Section 3.4 that the empirical properties of these tests are nearly identical across a wide range of alternatives including structural breaks, unit roots, and fractional integration.

The results also identify an important pitfall. It is quite misleading to conclude that a process is a unit root if you reject using the KPSS test, but that it is a structural change if you reject using the algebraically equivalent Andrews and Ploberger formulation. The best you can say with either test is that, if you reject the null of short memory, then the process is not short memory.

We will organize our discussion as follows. We first illustrate in Section 3.2 the connections between tests of structural change and nonstationarity and propose two new tests that fill in the unoccupied niches of the Andrews and Ploberger type statistics. Section 3.3 provides critical values and Section 3.4 provides evidence on the size and power of the tests. We summarize our results in Section 3.5.

3.2 Tests

3.2.1 The Direct Approach

The direct approach to testing using the null of short memory and the alternative of long memory is to estimate the long-run variance. We implement this idea by dividing a variance estimate that is sensitive to the presence or absence of long memory by a variance estimate that is less sensitive. Given a process y_t and the demeaned residuals $\epsilon_t = (y_t - \bar{y}_T)$, we propose the statistic

$$\lambda = \frac{s^2(T)}{s^2(\ell)},$$

where $s^2(T)$ and $s^2(\ell)$ are estimates of the long-run variance. The denominator is the familiar Newey and West (1987) kernel-based estimate $s^2(\ell)$ with bandwidth ℓ:

$$s^2(\ell) = T^{-1}\sum_{t=1}^{T}\epsilon_t^2 + 2T^{-1}\sum_{s=1}^{\ell} w(s,\ell)\sum_{t=s+1}^{T}\epsilon_t\epsilon_{t-s},$$

where

$$\epsilon_t = y_t - T^{-1}\sum_{t=1}^{T} y_t.$$

We will use the Bartlett window $w(s,\ell) = 1 - s/(\ell+1)$. A condition for consistency and positive definiteness is that $\ell \to \infty$ as $T \to \infty$, but that $\ell/T \to 0$. Typically $\ell \sim o_p(T^{1/2})$ is sufficient for this purpose. We will refer to $s^2(\ell)$ as a restricted-bandwidth kernel-based estimator. The numerator $s^2(T)$, which we will refer to as the maximum-bandwidth kernel-based estimator, uses the same kernel, but sets the bandwidth equal to the sample size.[2]

An intuitive argument illustrates why λ is a consistent test for long memory. If y_t is short memory, then the denominator $s^2(\ell)$ is a consistent estimate of the long run variance σ^2 and, while $s^2(T)$ is an inconsistent estimator for σ^2, it does converge in probability to a limiting distribution. If y_t is long memory, then $s^2(T)$ increases without limit as $T \to \infty$ because it is sensitive to the higher-order autocovariances and under long memory those autocovariances go to zero very slowly. In fact, the sum of the absolute values of the autocorrelations of y_t goes to infinity as $T \to \infty$. For $s^2(\ell)$, we distinguish two cases. If $\ell = 0$, it either converges to the variance of the process, if that variance is finite, or goes to infinity if it is not. For $\ell > 0$, such that $\ell/T \to 0$ as $T \to \infty$, we show that $s^2(\ell)$ goes to infinity, but more slowly than $s^2(T)$ as a result of fewer included autocovariances. Thus, the statistic is a way to measure the relative importance of higher order autocovariances and hence the likelihood of long run dependence.

While the ratio $s^2(T)/s^2(\ell)$ of long-run variance estimates is new, this statistic is not. We identify its place in the unit root testing literature using the following result.

[2] Kiefer and Vogelsang (2000) suggest the estimator $s^2(T)$ for use in HAC robust testing.

Proposition 1. *Given a process y_t and residuals $e_t = (y_t - \bar{y}_T)$, if $s^2(T)$ is computed using a Bartlett window, then*

$$s^2(T) = 2T^{-2}\sum_{t=1}^{T} S_t^2$$

where

$$S_t = \sum_{t=1}^{T} e_t$$

Proof. We will use the summations $S_t = \sum_{s=1}^{t} e_s$ for $t = 1, \ldots, T$ and $S_t^* = \sum_{s=t+1}^{T} e_s$ for $t = 1, \ldots, T$. Because these are residuals that sum to zero over the full sample, $S_t + S_t^* = 0$ for $t = 1, \ldots, T-1$. It is also the case that $S_T = 0$ and $S_0^* = 0$. Therefore,

$$\sum_{t=1}^{T-1} S_t^2 + \sum_{t=1}^{T-1} S_t^{*2} = 2\sum_{t=1}^{T} S_t^2$$

It will prove convenient to study

$$\sum_{t=1}^{T-1} S_t^2 = \sum_{t=1}^{T-1}(T-t)e_t^2 + 2\sum_{s=1}^{T-1}\sum_{t=s+1}^{T-1}(T-t)e_t e_{t-s}$$

Similarly,

$$\sum_{t=1}^{T-1} S_t^{*2} = \sum_{t=2}^{T}(t-1)e_t^2 + 2\sum_{s=1}^{T-1}\sum_{t=s+1}^{T}(t-s-1)e_t e_{t-s}$$

Adding these yields

$$2\sum_{t=1}^{T} S_t^2 = T\sum_{t=1}^{T} e_t^2 + 2\sum_{s=1}^{T-1}(T-s)\sum_{t=s+1}^{T} e_t e_{t-s}$$

Dividing both sides by T^2 gives

$$2T^{-2}\sum_{t=1}^{T} S_t^2 = T^{-1}\sum_{t=1}^{T} e_t^2 + 2T^{-1}\sum_{s=1}^{T-1}(1 - s/T)\sum_{t=s+1}^{T} e_t e_{t-s}$$

which proves the result ☐

This result shows that the statistic λ is twice the KPSS test statistic

$$\hat{\eta}_\mu = \frac{T^{-2}\sum_{t=1}^{T} S_t^2}{s^2(t)}$$

The distribution and power for various alternatives have been widely studied in the literature. Kwiatkowski et al. (1992) show the consistency of λ for a unit root. Lee and Schmidt (1996) find that λ is consistent for $I(d)$ processes where $d \in (0, 1/2)$, while Lee and Amsler (1997) show that this also holds for $I(d)$ for $d \in (1/2, 1)$. Andrews and Ploberger (1994) show the consistency of a statistic similar to λ (given in equation (3.6)) for a one time structural change.

3.2.2 Weighted Averages of LM Tests

A more general test based on the partial sums S_t can be written as:

$$\frac{T^{-2}\sum_{t=1}^{T} u_t S_t^2}{s^2(t)}$$

The observation that $E([t(T-t)]^{-1}S_t^2) = \sigma^2$ for i.i.d. y_t motivates the statistic

$$avgLM = T^{-2}\sum_{t=1}^{T}\frac{S_t^2}{t(T-t)s^2(t)} \tag{3.8}$$

This statistic is well known in the literature on testing for structural breaks. To establish this point, we use the following lemma:

Lemma 1. *The LM test for the alternative $y_t = \mu_0 + \varepsilon_t + u_t$, where ε_t is a short memory process and u_t is a structural break process that equals 0 for $t \in [0, \pi T)$ and β for $t \in [\pi T, T]$ (assuming one break at time πT), is given by*

$$LM_T(\pi) = \frac{S_{[\pi T]}^2}{T\pi(1-\pi)s^2(t)}$$

Proof. The LM version of the Andrews test requires only the full-sample GMM estimate. Using Andrews' notation, the statistic is given in equation (4.4) on page 836 of Andrews (1993).

$$LM_t(\pi) \simeq \frac{T}{\pi(1-\pi)} m_{1T}(\hat{\theta}, \pi)'\hat{S}^{-1}\hat{M}(\hat{M}'\hat{S}^{-1}\hat{M})'\hat{M}'\hat{S}^{-1}m_{1T}(\hat{\theta}, \pi) \tag{3.9}$$

where $m_{1T}(\hat{\theta}, \pi) = \frac{1}{T}\sum_{t=1}^{T\pi} m(W_t, \hat{\beta}, \hat{\delta})$, and where $\hat{\beta}$ and $\hat{\delta}$ are the full-sample estimates with the δ's being fixed parameters (i.e. the ones that do not have a structural change) and W_t is the observed data. Here, the estimator $\hat{S}$ is a kernel estimator of the spectral density matrix at frequency zero of the sequence of random variables $\{m(W_t, \beta_0, \delta_0) : t \leq T\}$. $\Pi = [\pi_0, 1-\pi_0]$ for $0 < \pi_0 < 1$.

$$\hat{M} = \frac{1}{T}\sum_{t=1}^{T}\frac{\partial m(W_t, \hat{\beta}, \hat{\delta})}{\partial \beta'} \tag{3.10}$$

For no temporal dependence $\hat{S}$ is given by the following equation.

$$\hat{S} = \frac{1}{T}\sum_{t=1}^{T}\left(m(W_t, \hat{\beta}, \hat{\delta}) - m_T\right)\left(m(W_t, \hat{\beta}, \hat{\delta}) - m_T\right)' \tag{3.11}$$

With temporal dependence, $\hat{S}$ is given by the slightly more complicated form.

$$\begin{aligned}\hat{S} &= \sum_{\nu=0}^{T-1} w(\nu/\ell(T)) \\ &\times \frac{1}{T}\sum_{t=\nu+1}^{T}\left(m(W_t,\hat{\beta},\hat{\delta}) - m_T\right)\left(m(W_{t-\nu},\hat{\beta},\hat{\delta}) - m_T\right)' \\ &+ \sum_{\nu=1}^{T-1} w(\nu/\ell(T)) \\ &\times \frac{1}{T}\sum_{t=\nu+1}^{T}\left(m(W_{t-\nu},\hat{\beta},\hat{\delta}) - m_T\right)\left(m(W_t,\hat{\beta},\hat{\delta}) - m_T\right)'\end{aligned} \tag{3.12}$$

where $w(\nu/\ell(T))$ is a kernel with bandwidth $\ell(T)$.

For the simple case of a change in mean under the null hypothesis of no structural break, the variable y_t is generated by the simple model

$$y_t = \mu_0 + \epsilon_t \qquad t = 1,\ldots,T. \tag{3.13}$$

The moment conditions evaluated at μ_0 will be

$$m(y_t,\mu_0) = E(\epsilon_t) = E(y_t - \mu_0) = 0 \tag{3.14}$$

The full sample estimate of μ_0 will be

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} y_t$$

Thus the sample moment condition will be

$$m(y_t,\hat{\mu}) = E(\hat{\epsilon}) = E(y_t - \hat{\mu}) = 0$$

This yields

$$\frac{\partial m(y_t,\hat{\mu})}{\partial\hat{\mu}} = \frac{\partial E(y_t - \hat{\mu})}{\partial\hat{\mu}} = -1 \tag{3.15}$$

Furthermore, $m_T = 0$ by construction and

$$m_{1T} = \frac{1}{T}\sum_{t=1}^{\lfloor \pi T\rfloor}\hat{\epsilon}_t = \frac{1}{T}S_{\lfloor \pi T\rfloor} \tag{3.16}$$

where $S_{\lfloor \pi T\rfloor}$ is the cumulative sum of $\hat{\epsilon}_t$ from 1 to the integer part of πT. Direct substitution of equation (3.15) into equation (3.10) yields

$$\hat{M} = \frac{1}{T}\sum_{t=1}^{T}(-1) = \frac{1}{T}(-T) = -1. \tag{3.17}$$

If there is no temporal dependence then $\hat{S}$ simplifies to

$$\hat{S} = \frac{1}{T}\sum_{t=1}^{T}\hat{\epsilon}_t^2 = \hat{\sigma}^2.$$

For temporally dependent ϵ_t, $\hat{S}$ is given by

$$\hat{S} = T^{-1}\sum_{t=1}^{T}\hat{\epsilon}_t^2 + 2T^{-1}\sum_{\nu=1}^{T} w(\nu/\ell(T)) \sum_{t=\nu+1}^{T}\hat{\epsilon}_t\hat{\epsilon}_{t-s}. \qquad (3.18)$$

If $w(\nu/\ell(T))$ is taken to be the Bartlett kernel[3], equation (3.18) becomes the $s^2(\ell)$ denominator of the KPSS statistic.

Gathering (3.16), (3.17), and (3.18) and substituting into equation (3.9) we have

$$\mathrm{LM}_T(\pi) \simeq T^{-1}\frac{S^2_{[\pi T]}}{\pi(1-\pi)s^2(\ell)} \qquad (3.19)$$

□

We thus see that the test given in equation (3.8) is a weighted average of LM tests for a structural break at an unknown breakpoint. This statistic, proposed by Andrews and Ploberger (1994) and given in Section 3.1 as (3.6), uses the notation $J(\pi)$ for the weighting on the LM statistics. For the case of an unknown breakpoint, they recommend a uniform $J(\pi)$ weighting on the LM statistics. Lemma 1 shows that this corresponds to $[\pi(1-\pi)]^{-1}$ weighting on the S_t^2 statistics.

We can summarize the difference between the KPSS test for a unit root and the Andrews-Ploberger test for a structural break as the difference between $J(\pi) = \pi(1-\pi)$ for π between 0 and 1 and $J(\pi) = 1$ for π between π_0 and $1-\pi_0$. This relationship has been obliquely mentioned by Andrews et al. (1996) among others but is often overlooked in empirical applications. We consider evidence on the difference between these tests later, but we note here two points. First, a constant function is a fair approximation of $\pi(1-\pi)$ over the middle of the range between 0 and 1. Second, Andrews and Ploberger truncate the range of π to avoid the endpoints 0 and 1. Given these two observations, it would not be entirely surprising if the two tests had similar performances for various alternatives. Our later results confirm this conjecture.

[3] The Newey-West estimator of the long-run variance uses a Bartlett kernel

3.2.3 Nonlinear Functions of LM Tests

Andrews and Ploberger (1994) also propose taking a weighted average of a monotone nonlinear transformation of the LM statistic.

$$(1+c)^{-p/2}\int_{\pi\in\Pi}\exp\left(\frac{1}{2}\frac{c}{(1+c)}LM_T(\pi)\right)dJ(\pi). \tag{3.20}$$

They present results concerning the optimal choice of c for a given size break. As $c\rightarrow 0$, this statistic approaches the $avgLM$ statistic given in Eq. (3.8), and as $c\rightarrow\infty$, it approaches the normalized $expLM$ statistic.[4]

$$expLM=\log\int_{\pi=\pi_0}^{1-\pi_0}\exp\left(\frac{1}{2}LM_T(\pi)\right)d\pi \tag{3.21}$$

The new corresponding variation on the $avgS$ statistic is

$$expS=\log\int_{\pi=\pi_0}^{1-\pi_0}\exp\left(\frac{1}{2}LM_T(\pi)\right)\pi(1-\pi)d\pi \tag{3.22}$$

Andrews and Ploberger note that replacing $c/(1+c)$ in Eq. (3.20) with a parameter r and letting $r\rightarrow\infty$ produces, in the limit, the supremum of the LM statistics.

3.2.4 Supremums of LM Tests

Although Andrews (1993) only considers uniform weights, there is nothing about supremum-based tests that rules out alternative weighting schemes of the form $\sup_{\pi\in\Pi}LM_T(\pi)J(\pi)$. The Andrews test can be written as

$$supLM=\sup_{\pi\in\Pi}\frac{S_\pi^2}{\pi(1-\pi)s^2(\ell)} \tag{3.23}$$

where Π is an interval bounded away from the endpoints of $[0,1]$. The natural counterpart to the $avgS$ (KPSS) and $expS$ statistics uses $J(\pi)=\pi(1-\pi)$ to produce

$$supS=\sup_{\pi\in[0,1]}\frac{S_\pi^2}{s^2(\ell)}. \tag{3.24}$$

This new test is the supremum analog of the KPSS test. It puts more weight on tests for breaks that occur in the interior of the sample that the $supLM$ test.

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[4] Andrews and Ploberger use the notation $expLM$ to denote the general formula with c between 0 and ∞. We are using the notations $avgLM$ and $expLM$ to denote those two limiting cases.

3.3 Critical Values

In this section, we present an analytic formula for critical values for $supS$ as well as critical values for $expS$ calculated by simulation. We also add the case of multiple parameter changes ($p > 1$) to $avgS$. This generalizes the KPSS test into a test for structural change.

Andrews and Ploberger (1994) give critical values for $avgLM$ and $expLM$, while Andrews (1993) gives critical values for $supLM$. KPSS give critical values for $avgS$ where $p = 1$.

Under H_0, the asymptotics for all of these tests are based on

$$S_{[\pi T]} \rightarrow B_1(\pi) - \pi B_1(1)$$

where $B_1(\pi)$ is standard Brownian motion. $S_{[\pi T]}$, then, converges to a Brownian bridge. The distributions for the statistics are derived from

$$S^2_{[\pi T]} \xrightarrow{d} Q_1(\pi),$$

where $Q_1(\pi)$ is the square of a Brownian bridge. This generalizes for the case of p parameters changing at time π to

$$Q_p(\pi) = (B_p(\pi) - \pi B_p(1))'(B_p(\pi) - \pi B_p(1))$$

where $B_p(\pi)$ is a p-dimensional vector of independent Brownian motions. Then the distribution for LM-test of p parameter changes at time πT is given by

$$LM_T(\pi) \xrightarrow{d} Q_p(\pi)/(\pi(1-\pi)).$$

The asymptotic critical values for $supS = \sup\ s^2(T)/s^2(t)$ can be derived analytically. The test statistic is distributed as

$$supS \longrightarrow \sup_{\pi \in [0,1]} Q_p(\pi)$$

For the $p = 1$ case, we get a particularly simple approximation. Pitman and Yor (1999)[5] give

$$P(\sup S_t^2/\sigma^2 < b) = \sum_{n=-\infty}^{\infty} (-1)^n \exp(-2n^2 b^2)$$

We will use

$$P(\sup S_t^2/\sigma^2 < b) = \alpha \approx 1 - 2e^{-2b}.$$

where α is the desired significance level. The asymptotic critical values for $supS$ are given by (approximately)

$$b = -\frac{1}{2}\ln(\alpha/2).$$

For the case $p > 1$, Kiefer (1959) provides the distribution of $\sup_{[0\ 1]} Q_p(\pi)$.

Critical values for the new tests are provided in Tables 3.1–3.3. They are calculated using the same procedure as Andrews (1993) and Andrews and Ploberger (1994). The values reported in Tables 3.1–3.3 are estimates of the desired asymptotic critical values obtained by (i) approximating the distribution of the integrals over $[\pi_0, 1-\pi_0]$ in Eqs. (3.7), (3.22), and (3.5) by averages over a fine grid of points $\Pi(N)$ and (ii) simulating the resultant averages by Monte Carlo. The grid $\Pi(N)$ is defined by

$$\Pi(N) = [\pi_0, 1-\pi_0] \cup \{\pi = j/N : j = 0, \ldots, N\}$$

The value of N was chosen to be 3,600. Each realization from the asymptotic distribution of the discretized version of (3.7), (3.22), or (3.5) was obtained by simulating a p-vector $B_p(\cdot)$ of independent Brownian motions on [0,1] at the discrete points in $\Pi(N)$ and then computing the discrete average of the appropriate function of $(B_p(\pi) - \pi B_p(1))'(B_p(\pi) - \pi B_p(1))$. The number of repetitions R used was 10,000.

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[5] (See also Borodin and Salminen (1996). Their expression (1.15.8.1) directly yields

$$P(\sup S_t^2/\sigma^2 < b) = \sum_{k=-\infty}^{\infty} \exp(-2(2k)^2 b^2) - \exp(-2(2k+1)^2 b^2).$$

The proof that these two expression are equal is a simple matter of noting that the series $\{n^2\}$ has the same elements as the union of the series $\{(2k)^2\}$ and $\{(2k+1)^2\}$. For even moderately large b, both expressions are dominated by the three middle terms in

$$\cdots + \exp(-8b^2) - \exp(-2b^2) + 1 - \exp(-2b^2) + \exp(-8b^2) - \cdots$$

Although our tests do not require a trimming parameter as in Andrews and Ploberger, prior knowledge of the general location of break may be incorporated by choosing a π_0 as in their tests. Critical values for a several truncation parameters are included for each value of p for those researchers who are inclined to use them.

3.4 Empirical Properties

3.4.1 Break Sensitivity

The differences between the tests are most clearly seen by examining the impact on each statistic of a structural change at a point π for each point in the sample, which we call a statistic's "Break Sensitivity" function. Section 5.5 discusses these functions for various statistics in greater detail. Figure 3.1 presents the statistic values for a pure break at points π, normalized by dividing by the sample size for the six tests of interest as well as a test we label $AHBT$ for "*ad hoc* break test". For this test, we apply a standard Chow test for a structural break at the midpoint of the sample. We also consider *ad hoc* break tests for breaks at the 25th and 75th percentages of the sample.

We can summarize the limited differences among these tests as follows. The $avgS$, $expS$, and $supS$ statistics are more sensitive to breaks in the middle of the sample than are the corresponding $avgLM$, $expLM$, and $supLM$ statistics, which are more sensitive to breaks near the endpoints. We would expect more sensitivity in the middle to be helpful in detecting small structural changes that might only be detectable in the middle. The $avgLM$, $expLM$, and $supLM$ tests should, on the other hand, be better at detecting structural change near the endpoints. This could be important if the structural breaks are fairly large because all tests will detect a break in the middle of the sample and picking up a break near the endpoints will be an advantage.

3.4.2 Size

Tables 3.4 and 3.5 present the sizes of the tests for various short memory processes that fall under the null hypothesis. Specifically, we consider stationary AR(1) processes $y_y = y_{t-1} + \epsilon_t$, with autoregressive parameter ϕ ranging between 0 and 1, where $\epsilon_t \sim N(0, 1)$. The tables are the rejection probabilities from 1000 replications of the DGP for each value of ϕ. Particular attention is paid to the more persistent values of ϕ by looking at several values ($\phi = 0.90, 0.95, 0.99$) close to 1.

Overall, the tests have significant size distortions for stationary processes with strong short-run dynamics. This can be counteracted fairly effectively, particularly in large samples, by using a bandwidth correction in the estimation of the long-run variance. For the T=800 case with a bandwidth of $\ell(12)$, only the most persistent series where ϕ=0.99 still have poor size properties.

The S and LM tests perform comparably, with the S-tests having less size distortion than the LM_2-tests but slightly more than the LM_{15}-tests. Within the S and LM testing groups, the Avg tests have the least amount of size distortion. It is interesting to note that the *ad hoc* break test ($AHBT$) has the least size distortion, although it also has much lower power against the nonstationary case of ϕ=1.

3.4.3 Power

Tables 3.6 – 3.11 show rejection probabilities for alternatives of the form

$$y_t = \epsilon_t + r_t$$

where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ and r_t is one of four forms:

Case 1: Structural Break

$$r_t = \delta I_t, \qquad I_t = \begin{cases} 0 & t < \pi T \\ 1 & t \geq \pi T \end{cases}, \qquad \delta = 1, \ \pi \in [0, 1]$$

In the tables, BrkPt refers to π.

Case 2: Uniformly Distributed Structural Break

$$r_t = \delta I_t, \qquad I_t = \begin{cases} 0 & t < \pi T \\ 1 & t \geq \pi T \end{cases}, \qquad \pi \sim U[0, 1]$$

In the tables, BrkSize refers to δ/σ_ϵ.

Case 3: Unit Root

$$r_t = r_{t-1} + u_t, \qquad u_t \sim N(0, \sigma_u^2)$$

In the tables, σ refers to σ_u/σ_ϵ.

Case 4: Fractional Integration

$$(1-L)^d r_t = u_t, \qquad u_t \sim N(0, \sigma_u^2)$$

The dominant feature of these tables is the similarity of the tests. One surprising result is that the *ad hoc* break test which ignores all these enhancements does not fare as badly as one might expect. For example, in Table 3.4, the $AHBT_{50}$ test rejects a unit root with σ=0.2 about 84% of the time compared to 96% of the time for the KPSS/$avgS$ test (which is specifically designed with the unit root alternative in mind).

We can explain some of the small differences that do exist. Consider the KPSS/$avgS$ stationarity test, which uses $J(\pi) = \pi(1-\pi)$, and the Andrews-Ploberger $avgLM$ structural break test ($\pi_0 = 0.02$), which uses $J(\pi) = 1$. For both structural break alternatives and unit root alternatives, these tests are very nearly equivalent in power. The small differences that occur reflect the weighting. The KPSS test is marginally better at detecting small structural breaks and small unit roots because, with more weight in the middle of the sample period, the KPSS test is a little more likely to detect a change in the middle of the sample. Both tests are ineffective in detecting small changes near an endpoint.

This situation reverses for large breaks or strong unit roots. A large break in the middle of the sample is likely to be detected by either test, and the $avgLM$ test has the advantage of putting greater weight near the endpoints, making it more effective in detecting a break in that region. Comparing the $avgLM$ tests for $\pi_0 = 0.02$ and $\pi_0 = 0.15$ (given in the tables as aLM_2 and aLM_{15}, respectively) leads to the same conclusion. The larger truncation parameter makes the test more effective for small breaks and unit roots where σ_u/σ_ϵ is small and less effective for large breaks and unit roots where σ_u/σ_ϵ is large.

For all of these tests, increasing the bandwidth of the long-run variance estimator in the denominator reduces the power of the test. The loss of power is not particularly bad and becomes less severe as the sample size increases. The exception is the AHBT. As noted in above, it suffers from significant power reduction when short-run dynamics are taken into account in estimating the long-run variance.

Compared to the Avg tests, the Exp and Sup versions add a little power in the cases of large structural breaks, large unit roots, and fractional integration. In particular, the Exp

versions of the tests dominate the others in terms of power. In the few cases in which it is not the most powerful (e.g. very small structural breaks), its power is only slightly less than that of the *Avg*-statistics.

3.5 Summary and Conclusions

If you evaluate differences in terms of testing outcomes, then there is very little practical difference between structural change and other forms of nonstationarity such as unit roots and fractional integration. The Andrews test for structural change ($supLM$) and the KPSS test ($avgS$) are virtually identical in their powers for both types of alternatives. In fact, there are cases where the power of the Andrews test is greater than the power of the KPSS test when the alternative is a unit root. There are also cases where the power of the KPSS test is greater than the power of the Andrews test when the alternative is a structural break.

We decompose what small differences there are into two test attributes. We show that these tests differ because the KPSS test uses S-weighting, which weights the squared partials sums S_t^2 equally, while the Andrews test uses LM-weighting, which keeps the significance level for each LM test at α. They also differ because the KPSS test uses an average and the Andrews test uses a supremum. Following Andrews and Ploberger, we add an exponential-average transformation that is intermediate between the average and the supremum.

There is virtually no difference between LM-weighting, and S-weighting. Equivalently, there is virtually no difference between the KPSS test and the Andrews-Ploberger $avgLM$ test. There is also very little practical difference between the Andrews $supLM$ test for structural breaks and the test we propose based on the $supS$. The differences among the average, exponential-average, and supremum tests are a little bigger, but still very small.

It is a bit misleading, therefore, to identify the Andrews-Ploberger $avgLM$ test as a test for structural breaks and to identify the KPSS test as a test for nonstationarity or, in particular, unit roots and fractional integration. There are certainly no grounds for using any of these tests to distinguish between, for example, structural change and fractional integration.

Table 3.1: *AvgS* Critical Values

	p = 1			p = 2			p = 3			p = 4			p = 5		
π_0	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
0.49	0.69	0.96	1.65	1.16	1.49	2.38	1.58	1.96	2.85	1.98	2.41	3.37	2.30	2.75	3.90
0.48	0.68	0.95	1.61	1.14	1.47	2.30	1.55	1.93	2.80	1.94	2.37	3.31	2.26	2.71	3.76
0.47	0.67	0.94	1.58	1.13	1.47	2.29	1.54	1.90	2.76	1.93	2.36	3.27	2.25	2.68	3.75
0.45	0.67	0.93	1.54	1.12	1.44	2.22	1.52	1.86	2.68	1.90	2.32	3.24	2.21	2.63	3.63
0.40	0.64	0.88	1.45	1.07	1.38	2.09	1.46	1.77	2.51	1.84	2.23	3.13	2.12	2.54	3.42
0.35	0.60	0.82	1.35	1.01	1.31	1.92	1.39	1.69	2.39	1.76	2.12	2.96	2.04	2.44	3.21
0.30	0.56	0.78	1.27	0.96	1.24	1.81	1.31	1.59	2.23	1.68	2.01	2.75	1.96	2.30	3.05
0.25	0.52	0.73	1.20	0.91	1.15	1.67	1.23	1.51	2.09	1.59	1.89	2.56	1.86	2.18	2.86
0.20	0.49	0.67	1.10	0.85	1.08	1.56	1.16	1.41	1.94	1.50	1.77	2.39	1.75	2.04	2.66
0.15	0.45	0.61	1.00	0.79	0.99	1.44	1.09	1.31	1.79	1.40	1.64	2.20	1.64	1.90	2.47
0.10	0.42	0.57	0.91	0.73	0.91	1.31	1.01	1.20	1.65	1.29	1.52	2.02	1.53	1.75	2.27
0.05	0.38	0.51	0.82	0.67	0.84	1.19	0.92	1.10	1.51	1.18	1.39	1.83	1.40	1.61	2.08
0.02	0.36	0.48	0.77	0.63	0.79	1.12	0.87	1.03	1.42	1.12	1.31	1.73	1.33	1.52	1.96
0.00	0.35	0.47	0.74	0.61	0.76	1.08	0.84	0.99	1.37	1.08	1.26	1.66	1.28	1.46	1.88

	p = 6			p = 7			p = 8			p = 9			p = 10		
π_0	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
0.49	2.68	3.17	4.20	3.04	3.52	4.70	3.38	3.90	5.06	3.68	4.24	5.36	4.06	4.61	5.83
0.48	2.65	3.14	4.13	2.99	3.50	4.59	3.35	3.86	4.98	3.66	4.19	5.28	4.01	4.56	5.74
0.47	2.62	3.10	4.08	2.97	3.46	4.55	3.31	3.82	4.94	3.63	4.17	5.22	3.97	4.51	5.68
0.45	2.58	3.05	3.98	2.93	3.40	4.49	3.27	3.78	4.83	3.58	4.10	5.18	3.92	4.45	5.59
0.40	2.49	2.93	3.78	2.82	3.25	4.20	3.16	3.64	4.58	3.47	3.92	4.95	3.79	4.28	5.25
0.35	2.40	2.80	3.61	2.70	3.09	3.98	3.03	3.49	4.37	3.34	3.77	4.74	3.65	4.11	5.08
0.30	2.29	2.65	3.36	2.58	2.95	3.78	2.90	3.32	4.11	3.19	3.61	4.45	3.51	3.91	4.80
0.25	2.18	2.52	3.17	2.45	2.80	3.55	2.77	3.15	3.89	3.03	3.41	4.20	3.35	3.70	4.49
0.20	2.06	2.36	2.96	2.33	2.63	3.30	2.63	2.96	3.63	2.86	3.20	3.94	3.15	3.51	4.19
0.15	1.93	2.20	2.78	2.18	2.45	3.06	2.46	2.77	3.36	2.68	3.00	3.67	2.96	3.28	3.89
0.10	1.79	2.03	2.53	2.02	2.27	2.81	2.28	2.56	3.12	2.50	2.79	3.37	2.74	3.04	3.58
0.05	1.64	1.86	2.32	1.86	2.08	2.56	2.09	2.35	2.85	2.30	2.56	3.10	2.53	2.78	3.28
0.02	1.56	1.76	2.18	1.76	1.97	2.41	1.98	2.22	2.69	2.17	2.42	2.92	2.39	2.63	3.10
0.00	1.50	1.69	2.10	1.69	1.89	2.32	1.90	2.14	2.59	2.09	2.33	2.81	2.30	2.53	2.97

	p = 11			p = 12			p = 13			p = 14			p = 15		
π_0	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
0.49	4.34	4.94	6.25	4.64	5.23	6.59	5.03	5.66	6.96	5.26	5.92	7.37	5.63	6.31	7.73
0.48	4.29	4.87	6.13	4.59	5.16	6.50	4.96	5.59	6.85	5.20	5.84	7.25	5.55	6.24	7.65
0.47	4.27	4.84	6.10	4.56	5.12	6.44	4.91	5.55	6.84	5.17	5.81	7.19	5.50	6.18	7.55
0.45	4.20	4.76	5.99	4.50	5.05	6.33	4.85	5.47	6.66	5.10	5.74	7.07	5.45	6.09	7.44
0.40	4.07	4.62	5.74	4.38	4.92	6.04	4.73	5.30	6.36	4.96	5.56	6.76	5.32	5.89	7.09
0.35	3.93	4.42	5.46	4.22	4.72	5.77	4.59	5.09	6.07	4.77	5.37	6.45	5.15	5.69	6.74
0.30	3.78	4.21	5.16	4.05	4.51	5.51	4.40	4.88	5.75	4.59	5.13	6.16	4.94	5.41	6.40
0.25	3.60	4.00	4.89	3.87	4.27	5.22	4.20	4.62	5.45	4.38	4.86	5.86	4.71	5.16	6.07
0.20	3.41	3.79	4.56	3.67	4.04	4.87	3.97	4.35	5.15	4.16	4.60	5.49	4.47	4.87	5.71
0.15	3.20	3.54	4.24	3.44	3.79	4.56	3.73	4.08	4.79	3.91	4.32	5.12	4.20	4.57	5.31
0.10	2.97	3.28	3.90	3.20	3.51	4.19	3.47	3.79	4.41	3.65	4.02	4.70	3.92	4.26	4.93
0.05	2.74	3.02	3.58	2.95	3.22	3.83	3.20	3.47	4.03	3.37	3.69	4.31	3.61	3.91	4.53
0.02	2.59	2.86	3.38	2.79	3.05	3.62	3.03	3.28	3.82	3.19	3.49	4.07	3.42	3.70	4.28
0.00	2.49	2.75	3.25	2.68	2.93	3.48	2.91	3.16	3.67	3.06	3.36	3.91	3.29	3.56	4.12

	p = 16			p = 17			p = 18			p = 19			p = 20		
π_0	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
0.49	5.96	6.59	7.34	6.26	6.98	8.31	6.53	7.29	8.77	6.87	7.59	9.16	7.18	7.98	9.47
0.48	5.88	6.50	7.77	6.21	6.89	8.14	6.46	7.23	8.69	6.80	7.53	9.01	7.10	7.90	9.37
0.47	5.84	6.45	7.72	6.18	6.83	8.09	6.43	7.17	8.60	6.76	7.49	8.90	7.07	7.83	9.35
0.45	5.76	6.39	7.55	6.08	6.73	7.96	6.35	7.05	8.47	6.68	7.38	8.73	7.01	7.70	9.21
0.40	5.61	6.17	7.27	5.92	6.54	7.68	6.15	6.80	8.20	6.48	7.12	8.43	6.79	7.45	8.79
0.35	5.42	5.91	6.96	5.74	6.28	7.35	5.97	6.56	7.86	6.29	6.84	8.06	6.57	7.21	8.43
0.30	5.21	5.67	6.65	5.53	6.02	7.02	5.75	6.27	7.43	6.03	6.59	7.74	6.32	6.90	8.04
0.25	4.97	5.42	6.30	5.29	5.74	6.63	5.48	5.97	7.06	5.77	6.27	7.33	6.05	6.57	7.64
0.20	4.72	5.12	5.92	5.01	5.43	6.30	5.20	5.66	6.64	5.46	5.93	6.87	5.72	6.20	7.20
0.15	4.44	4.80	5.51	4.71	5.10	5.89	4.90	5.30	6.18	5.13	5.56	6.40	5.39	5.81	6.73
0.10	4.13	4.47	5.09	4.38	4.75	5.45	4.56	4.92	5.73	4.79	5.16	5.93	5.01	5.39	6.24
0.05	3.81	4.11	4.68	4.04	4.37	5.00	4.20	4.53	5.27	4.42	4.76	5.44	4.62	4.97	5.74
0.02	3.61	3.89	4.43	3.82	4.13	4.73	3.98	4.28	4.98	4.18	4.50	5.14	4.38	4.70	5.44
0.00	3.47	3.74	4.25	3.68	3.97	4.55	3.83	4.12	4.79	4.02	4.33	4.94	4.21	4.52	5.23

Table 3.2: *ExpS* Critical Values

π_0	p = 1 10%	p = 1 5%	p = 1 1%	p = 2 10%	p = 2 5%	p = 2 1%	p = 3 10%	p = 3 5%	p = 3 1%	p = 4 10%	p = 4 5%	p = 4 1%	p = 5 10%	p = 5 5%	p = 5 1%
0.49	1.79	2.35	3.69	2.73	3.40	5.18	3.59	4.35	6.12	4.36	5.23	7.15	5.02	5.90	8.20
0.48	1.79	2.35	3.69	2.74	3.43	5.15	3.58	4.32	6.17	4.36	5.24	7.17	5.02	5.98	8.16
0.47	1.81	2.35	3.65	2.76	3.43	5.12	3.61	4.35	6.18	4.40	5.29	7.19	5.04	5.98	8.18
0.45	1.82	2.36	3.66	2.77	3.48	5.14	3.64	4.36	6.24	4.45	5.32	7.24	5.07	6.03	8.14
0.40	1.83	2.37	3.65	2.82	3.52	5.21	3.67	4.44	6.22	4.53	5.40	7.55	5.19	6.19	8.08
0.35	1.82	2.38	3.75	2.83	3.53	5.16	3.73	4.47	6.27	4.57	5.43	7.47	5.27	6.24	8.15
0.30	1.80	2.36	3.70	2.83	3.54	5.17	3.73	4.44	6.18	4.59	5.44	7.56	5.35	6.26	8.17
0.25	1.78	2.33	3.77	2.83	3.54	5.11	3.72	4.46	6.20	4.62	5.49	7.45	5.42	6.25	8.19
0.20	1.75	2.29	3.70	2.81	3.53	5.15	3.72	4.44	6.28	4.65	5.50	7.43	5.45	6.30	8.23
0.15	1.72	2.24	3.63	2.78	3.48	5.04	3.70	4.44	6.23	4.66	5.48	7.37	5.46	6.27	8.13
0.10	1.68	2.20	3.57	2.74	3.43	5.04	3.65	4.40	6.20	4.63	5.43	7.37	5.44	6.27	8.14
0.05	1.60	2.12	3.50	2.69	3.39	4.99	3.60	4.35	6.12	4.57	5.42	7.28	5.41	6.25	8.07
0.02	1.55	2.08	3.44	2.64	3.35	4.96	3.55	4.31	6.05	4.54	5.40	7.25	5.37	6.20	8.01
0.00	1.52	2.04	3.40	2.60	3.31	4.92	3.52	4.27	6.02	4.51	5.37	7.21	5.33	6.18	7.97

π_0	p = 6 10%	p = 6 5%	p = 6 1%	p = 7 10%	p = 7 5%	p = 7 1%	p = 8 10%	p = 8 5%	p = 8 1%	p = 9 10%	p = 9 5%	p = 9 1%	p = 10 10%	p = 10 5%	p = 10 1%
0.49	5.78	6.76	8.84	6.48	7.45	9.75	7.20	8.23	10.58	7.80	8.89	11.12	8.52	9.67	12.03
0.48	5.80	6.79	8.81	6.50	7.53	9.78	7.23	8.28	10.58	7.89	8.98	11.11	8.56	9.71	12.10
0.47	5.82	6.81	8.84	6.57	7.56	9.84	7.27	8.35	10.61	7.92	9.04	11.10	8.59	9.73	12.14
0.45	5.90	6.85	8.87	6.62	7.62	9.86	7.33	8.45	10.70	8.01	9.11	11.23	8.75	9.84	12.22
0.40	6.01	6.98	8.97	6.72	7.71	9.79	7.49	8.57	10.65	8.16	9.19	11.46	8.93	10.01	12.31
0.35	6.07	7.05	8.99	6.81	7.75	9.98	7.61	8.70	10.74	8.27	9.28	11.69	9.04	10.11	12.47
0.30	6.12	7.02	8.99	6.89	7.79	9.98	7.71	8.83	10.80	8.36	9.37	11.69	9.11	10.19	12.49
0.25	6.17	7.08	9.00	6.91	7.86	10.02	7.78	8.87	10.93	8.41	9.45	11.72	9.18	10.29	12.53
0.20	6.21	7.09	9.02	6.96	7.92	10.06	7.83	8.84	10.87	8.45	9.48	11.72	9.25	10.34	12.56
0.15	6.23	7.10	9.01	6.99	7.93	9.99	7.88	8.82	10.82	8.51	9.59	11.68	9.32	10.34	12.58
0.10	6.22	7.12	9.01	6.97	7.94	9.93	7.91	8.84	10.76	8.52	9.62	11.64	9.34	10.39	12.56
0.05	6.20	7.09	8.97	6.96	7.90	9.84	7.87	8.85	10.71	8.52	9.61	11.58	9.32	10.36	12.50
0.02	6.17	7.05	8.94	6.95	7.85	9.79	7.86	8.83	10.65	8.49	9.56	11.60	9.30	10.33	12.52
0.00	6.14	7.02	8.90	6.93	7.81	9.76	7.83	8.81	10.63	8.46	9.55	11.60	9.27	10.31	12.50

π_0	p = 11 10%	p = 11 5%	p = 11 1%	p = 12 10%	p = 12 5%	p = 12 1%	p = 13 10%	p = 13 5%	p = 13 1%	p = 14 10%	p = 14 5%	p = 14 1%	p = 15 10%	p = 15 5%	p = 15 1%
0.49	9.09	10.30	12.87	9.68	10.86	13.57	10.47	11.72	14.32	10.92	12.24	15.17	11.67	13.03	15.91
0.48	9.14	10.33	12.92	9.77	10.90	13.62	10.48	11.82	14.46	11.01	12.28	15.20	11.71	13.14	15.91
0.47	9.19	10.41	12.95	9.85	10.99	13.77	10.54	11.89	14.52	11.09	12.42	15.21	11.76	13.20	15.97
0.45	9.28	10.51	13.05	9.93	11.10	13.74	10.64	11.99	14.47	11.17	12.60	15.37	11.93	13.31	16.12
0.40	9.49	10.70	13.47	10.10	11.37	13.85	10.97	12.26	14.74	11.46	12.86	15.41	12.29	13.62	16.22
0.35	9.65	10.81	13.55	10.31	11.57	14.01	11.20	12.47	14.88	11.63	13.01	15.69	12.50	13.79	16.48
0.30	9.77	10.90	13.55	10.48	11.72	14.25	11.34	12.58	15.01	11.78	13.13	15.88	12.64	13.90	16.57
0.25	9.86	10.98	13.51	10.60	11.78	14.34	11.48	12.61	15.08	11.93	13.25	15.88	12.77	13.99	16.67
0.20	9.93	11.02	13.57	10.70	11.84	14.33	11.59	12.68	15.12	12.04	13.28	16.00	12.87	14.09	16.73
0.15	10.00	11.05	13.56	10.77	11.89	14.29	11.63	12.74	15.18	12.13	13.34	16.04	12.93	14.16	16.74
0.10	10.06	11.10	13.56	10.82	11.94	14.19	11.67	12.78	15.38	12.21	13.39	16.01	12.99	14.19	16.72
0.05	10.07	11.09	13.52	10.82	11.91	14.11	11.68	12.77	15.32	12.24	13.41	15.95	13.02	14.22	16.74
0.02	10.09	11.07	13.49	10.84	11.90	14.08	11.69	12.75	15.34	12.24	13.41	15.98	13.01	14.23	16.70
0.00	10.06	11.04	13.48	10.81	11.89	14.05	11.66	12.71	15.30	12.24	13.38	15.97	12.99	14.22	16.69

π_0	p = 16 10%	p = 16 5%	p = 16 1%	p = 17 10%	p = 17 5%	p = 17 1%	p = 18 10%	p = 18 5%	p = 18 1%	p = 19 10%	p = 19 5%	p = 19 1%	p = 20 10%	p = 20 5%	p = 20 1%
0.49	12.32	13.60	16.10	12.94	14.36	17.02	13.49	14.99	17.90	14.12	15.61	18.78	14.77	16.35	19.42
0.48	12.38	13.66	16.25	13.06	14.46	17.03	13.57	15.10	18.07	14.24	15.75	18.76	14.87	16.47	19.53
0.47	12.47	13.78	16.31	13.15	14.50	17.12	13.68	15.16	18.11	14.34	15.82	18.71	14.99	16.61	19.67
0.45	12.59	13.94	16.52	13.31	14.61	17.26	13.87	15.29	18.33	14.50	15.95	18.79	15.19	16.75	19.69
0.40	12.90	14.18	16.72	13.56	14.89	17.47	14.12	15.49	18.65	14.80	16.21	19.16	15.50	17.03	19.83
0.35	13.12	14.39	16.92	13.76	15.09	17.82	14.39	15.76	18.68	15.06	16.44	19.21	15.75	17.26	20.19
0.30	13.28	14.52	17.16	13.97	15.35	17.98	14.53	15.99	18.85	15.24	16.66	19.31	15.97	17.43	20.25
0.25	13.40	14.73	17.39	14.17	15.53	18.16	14.70	16.06	18.99	15.37	16.83	19.57	16.09	17.58	20.44
0.20	13.48	14.78	17.42	14.29	15.64	18.28	14.84	16.20	19.03	15.57	16.94	19.82	16.23	17.70	20.50
0.15	13.55	14.80	17.43	14.38	15.72	18.31	14.94	16.27	19.09	15.68	16.99	19.85	16.30	17.79	20.64
0.10	13.60	14.86	17.38	14.41	15.77	18.32	15.06	16.37	19.02	15.77	17.16	19.80	16.41	17.79	20.65
0.05	13.68	14.91	17.34	14.45	15.80	18.42	15.12	16.41	18.97	15.85	17.15	19.80	16.47	17.84	20.75
0.02	13.70	14.92	17.34	14.49	15.80	18.36	15.13	16.46	19.05	15.88	17.15	19.80	16.52	17.82	20.72
0.00	13.69	14.90	17.30	14.48	15.77	18.31	15.13	16.44	19.06	15.89	17.13	19.80	16.53	17.81	20.72

Table 3.3: *SupS* Critical Values

π_0	p = 1			p = 2			p = 3			p = 4			p = 5		
	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
0.49	0.85	1.16	1.88	1.36	1.74	2.67	1.83	2.24	3.19	2.24	2.72	3.71	2.58	3.07	4.21
0.48	0.93	1.25	1.98	1.47	1.86	2.80	1.94	2.35	3.34	2.38	2.83	3.86	2.70	3.23	4.36
0.47	1.00	1.31	2.07	1.54	1.96	2.87	2.01	2.42	3.50	2.47	2.94	3.98	2.81	3.34	4.44
0.45	1.09	1.42	2.17	1.66	2.07	2.99	2.14	2.56	3.64	2.61	3.08	4.15	2.95	3.49	4.57
0.40	1.24	1.60	2.38	1.84	2.25	3.24	2.34	2.79	3.81	2.83	3.32	4.53	3.21	3.74	4.77
0.35	1.35	1.71	2.52	1.96	2.36	3.32	2.45	2.89	3.89	2.94	3.42	4.61	3.34	3.87	4.91
0.30	1.42	1.76	2.57	2.03	2.44	3.37	2.53	2.95	3.93	3.01	3.50	4.64	3.41	3.94	4.95
0.25	1.45	1.78	2.63	2.07	2.48	3.40	2.56	2.99	3.97	3.05	3.53	4.66	3.45	3.95	4.96
0.20	1.48	1.80	2.65	2.08	2.50	3.40	2.58	3.01	3.98	3.05	3.54	4.66	3.47	3.97	5.00
0.15	1.49	1.81	2.65	2.09	2.50	3.40	2.59	3.01	3.98	3.05	3.54	4.66	3.47	3.97	5.00
0.10	1.49	1.81	2.65	2.09	2.50	3.40	2.59	3.01	3.98	3.05	3.54	4.66	3.47	3.97	5.00
0.05	1.49	1.81	2.65	2.09	2.50	3.40	2.59	3.01	3.98	3.05	3.54	4.66	3.47	3.97	5.00
0.02	1.49	1.81	2.65	2.09	2.50	3.40	2.59	3.01	3.98	3.05	3.54	4.66	3.47	3.97	5.00
0.00	1.49	1.81	2.65	2.09	2.50	3.40	2.59	3.01	3.98	3.05	3.54	4.66	3.47	3.97	5.00

π_0	p = 6			p = 7			p = 8			p = 9			p = 10		
	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
0.49	2.98	3.49	4.59	3.35	3.87	5.10	3.74	4.28	5.50	4.06	4.63	5.77	4.42	5.03	6.26
0.48	3.13	3.66	4.73	3.50	4.04	5.23	3.89	4.46	5.63	4.24	4.79	5.95	4.59	5.22	6.46
0.47	3.24	3.76	4.78	3.62	4.16	5.33	4.01	4.57	5.80	4.35	4.92	6.03	4.72	5.33	6.59
0.45	3.41	3.91	5.01	3.76	4.33	5.50	4.18	4.77	5.97	4.54	5.12	6.26	4.94	5.50	6.78
0.40	3.63	4.16	5.26	4.04	4.55	5.68	4.44	5.02	6.16	4.79	5.36	6.54	5.22	5.81	7.05
0.35	3.76	4.29	5.34	4.17	4.68	5.81	4.58	5.14	6.23	4.93	5.47	6.72	5.33	5.92	7.15
0.30	3.82	4.34	5.39	4.24	4.73	5.86	4.65	5.23	6.27	5.00	5.52	6.75	5.40	5.97	7.18
0.25	3.85	4.36	5.39	4.27	4.77	5.86	4.67	5.26	6.32	5.03	5.53	6.75	5.42	5.99	7.18
0.20	3.86	4.37	5.39	4.28	4.77	5.87	4.69	5.26	6.32	5.03	5.54	6.75	5.42	6.00	7.18
0.15	3.86	4.37	5.39	4.28	4.78	5.87	4.69	5.26	6.32	5.03	5.54	6.75	5.42	6.00	7.18
0.10	3.86	4.37	5.39	4.28	4.78	5.87	4.69	5.26	6.32	5.03	5.54	6.75	5.42	6.00	7.18
0.05	3.86	4.37	5.39	4.28	4.78	5.87	4.69	5.26	6.32	5.03	5.54	6.75	5.42	6.00	7.18
0.02	3.86	4.37	5.39	4.28	4.78	5.87	4.69	5.26	6.32	5.03	5.54	6.75	5.42	6.00	7.18
0.00	3.86	4.37	5.39	4.28	4.78	5.87	4.69	5.26	6.32	5.03	5.54	6.75	5.42	6.00	7.18

π_0	p = 11			p = 12			p = 13			p = 14			p = 15		
	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
0.49	4.71	5.32	6.70	5.03	5.65	7.04	5.41	6.08	7.44	5.65	6.34	7.88	6.05	6.75	8.24
0.48	4.89	5.52	6.86	5.22	5.83	7.26	5.59	6.30	7.67	5.87	6.54	8.05	6.24	6.96	8.40
0.47	5.01	5.66	7.03	5.37	5.96	7.38	5.72	6.42	7.79	6.00	6.74	8.16	6.39	7.10	8.56
0.45	5.20	5.84	7.24	5.53	6.18	7.50	5.92	6.62	7.95	6.22	6.96	8.38	6.62	7.31	8.74
0.40	5.49	6.13	7.53	5.81	6.50	7.80	6.29	6.96	8.27	6.51	7.27	8.64	6.95	7.65	9.03
0.35	5.65	6.25	7.67	5.98	6.64	7.97	6.42	7.09	8.42	6.65	7.38	8.74	7.10	7.80	9.21
0.30	5.72	6.30	7.69	6.05	6.70	8.00	6.49	7.15	8.46	6.74	7.45	8.76	7.17	7.86	9.21
0.25	5.74	6.31	7.70	6.08	6.71	8.00	6.51	7.16	8.46	6.76	7.47	8.78	7.19	7.87	9.21
0.20	5.75	6.31	7.70	6.08	6.71	8.00	6.51	7.16	8.46	6.77	7.47	8.78	7.20	7.87	9.21
0.15	5.75	6.31	7.70	6.08	6.71	8.00	6.51	7.16	8.46	6.77	7.47	8.78	7.20	7.87	9.21
0.10	5.75	6.31	7.70	6.08	6.71	8.00	6.51	7.16	8.46	6.77	7.47	8.78	7.20	7.87	9.21
0.05	5.75	6.31	7.70	6.08	6.71	8.00	6.51	7.16	8.46	6.77	7.47	8.78	7.20	7.87	9.21
0.02	5.75	6.31	7.70	6.08	6.71	8.00	6.51	7.16	8.46	6.77	7.47	8.78	7.20	7.87	9.21
0.00	5.75	6.31	7.70	6.08	6.71	8.00	6.51	7.16	8.46	6.77	7.47	8.78	7.20	7.87	9.21

π_0	p = 16			p = 17			p = 18			p = 19			p = 20		
	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
0.49	6.36	7.03	8.38	6.73	7.44	8.82	6.97	7.75	9.29	7.32	8.08	9.69	7.66	8.44	10.03
0.48	6.57	7.27	8.61	6.94	7.64	9.02	7.20	7.97	9.57	7.53	8.30	9.89	7.89	8.71	10.28
0.47	6.70	7.43	8.79	7.07	7.79	9.14	7.35	8.12	9.74	7.69	8.50	10.01	8.06	8.90	10.47
0.45	6.92	7.65	8.99	7.33	7.98	9.43	7.60	8.37	9.93	7.92	8.71	10.18	8.29	9.10	10.72
0.40	7.28	7.96	9.32	7.59	8.32	9.77	7.92	8.62	10.20	8.25	9.03	10.49	8.62	9.42	10.89
0.35	7.43	8.11	9.40	7.75	8.46	9.85	8.08	8.78	10.27	8.42	9.17	10.63	8.79	9.57	11.05
0.30	7.51	8.16	9.46	7.83	8.52	9.87	8.14	8.83	10.33	8.48	9.23	10.65	8.84	9.60	11.11
0.25	7.52	8.17	9.47	7.85	8.55	9.87	8.16	8.84	10.34	8.50	9.25	10.68	8.86	9.62	11.11
0.20	7.52	8.17	9.47	7.86	8.55	9.87	8.16	8.84	10.34	8.50	9.25	10.68	8.86	9.62	11.11
0.15	7.52	8.17	9.47	7.86	8.55	9.87	8.16	8.84	10.34	8.50	9.25	10.68	8.86	9.62	11.11
0.10	7.52	8.17	9.47	7.86	8.55	9.87	8.16	8.84	10.34	8.50	9.25	10.68	8.86	9.62	11.11
0.05	7.52	8.17	9.47	7.86	8.55	9.87	8.16	8.84	10.34	8.50	9.25	10.68	8.86	9.62	11.11
0.02	7.52	8.17	9.47	7.86	8.55	9.87	8.16	8.84	10.34	8.50	9.25	10.68	8.86	9.62	11.11
0.00	7.52	8.17	9.47	7.86	8.55	9.87	8.16	8.84	10.34	8.50	9.25	10.68	8.86	9.62	11.11

Table 3.4: Size and Power: AR(1) Models, T=200

Bandwidth = $\ell(0)$

ρ	avgS	expS	supS	aLM$_2$	eLM$_2$	sLM$_2$	aLM$_{15}$	eLM$_{15}$	sLM$_{15}$	ah$_{25}$	ah$_{50}$	ah$_{75}$
0.00	0.056	0.056	0.045	0.055	0.055	0.044	0.054	0.055	0.042	0.051	0.056	0.054
0.10	0.073	0.082	0.076	0.075	0.085	0.072	0.074	0.083	0.076	0.067	0.064	0.084
0.50	0.335	0.497	0.432	0.377	0.512	0.489	0.332	0.442	0.447	0.257	0.262	0.247
0.80	0.819	0.941	0.909	0.865	0.942	0.923	0.796	0.898	0.908	0.494	0.542	0.496
0.85	0.900	0.972	0.958	0.938	0.974	0.970	0.886	0.948	0.954	0.555	0.551	0.596
0.90	0.973	0.993	0.985	0.984	0.993	0.990	0.968	0.983	0.987	0.653	0.660	0.655
0.95	0.992	0.998	0.997	0.996	0.998	0.997	0.989	0.997	0.997	0.715	0.763	0.749
0.99	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	0.807	0.874	0.835
1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.888	0.905	0.893

Bandwidth = $\ell(4)$

ρ	avgS	expS	supS	aLM$_2$	eLM$_2$	sLM$_2$	aLM$_{15}$	eLM$_{15}$	sLM$_{15}$	ah$_{25}$	ah$_{50}$	ah$_{75}$
0.00	0.035	0.037	0.026	0.035	0.035	0.026	0.038	0.036	0.024	0.050	0.041	0.032
0.10	0.045	0.043	0.035	0.047	0.041	0.028	0.051	0.039	0.033	0.051	0.045	0.061
0.50	0.085	0.085	0.072	0.078	0.082	0.043	0.084	0.086	0.058	0.082	0.084	0.084
0.80	0.217	0.280	0.222	0.225	0.265	0.169	0.231	0.277	0.219	0.171	0.196	0.213
0.85	0.312	0.391	0.314	0.317	0.371	0.264	0.318	0.385	0.336	0.229	0.241	0.254
0.90	0.437	0.528	0.443	0.450	0.523	0.388	0.450	0.518	0.464	0.307	0.336	0.309
0.95	0.641	0.738	0.646	0.665	0.730	0.621	0.649	0.724	0.682	0.380	0.492	0.456
0.99	0.863	0.905	0.842	0.889	0.915	0.840	0.865	0.891	0.866	0.568	0.675	0.602
1.00	0.932	0.953	0.913	0.940	0.954	0.909	0.928	0.945	0.921	0.721	0.794	0.731

Bandwidth = $\ell(12)$

ρ	avgS	expS	supS	aLM$_2$	eLM$_2$	sLM$_2$	aLM$_{15}$	eLM$_{15}$	sLM$_{15}$	ah$_{25}$	ah$_{50}$	ah$_{75}$
0.00	0.036	0.019	0.014	0.030	0.018	0.023	0.039	0.020	0.015	0.050	0.039	0.051
0.10	0.033	0.026	0.017	0.031	0.022	0.021	0.035	0.022	0.012	0.050	0.046	0.042
0.50	0.055	0.036	0.035	0.049	0.032	0.010	0.059	0.037	0.017	0.056	0.062	0.069
0.80	0.076	0.072	0.043	0.070	0.051	0.009	0.085	0.075	0.022	0.062	0.086	0.107
0.85	0.124	0.098	0.047	0.102	0.073	0.012	0.130	0.114	0.032	0.116	0.113	0.128
0.90	0.144	0.153	0.086	0.126	0.123	0.017	0.155	0.165	0.052	0.118	0.155	0.150
0.95	0.295	0.297	0.202	0.257	0.261	0.057	0.308	0.314	0.152	0.190	0.276	0.220
0.99	0.548	0.586	0.442	0.543	0.563	0.268	0.558	0.587	0.403	0.336	0.481	0.410
1.00	0.728	0.760	0.632	0.736	0.757	0.462	0.742	0.769	0.648	0.540	0.648	0.548

Table 3.5: Size and Power: AR(1) Models. T=800

Bandwidth = $\ell(0)$

ρ	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.00	0.052	0.054	0.049	0.055	0.054	0.051	0.054	0.056	0.051	0.051	0.051	0.057
0.10	0.068	0.076	0.079	0.063	0.071	0.072	0.066	0.071	0.075	0.061	0.070	0.067
0.50	0.354	0.563	0.530	0.411	0.572	0.642	0.351	0.479	0.533	0.258	0.255	0.276
0.80	0.864	0.976	0.960	0.939	0.982	0.986	0.809	0.934	0.955	0.529	0.524	0.511
0.85	0.938	0.995	0.979	0.971	0.995	0.995	0.908	0.970	0.980	0.582	0.580	0.568
0.90	0.985	1.000	0.999	0.994	1.000	1.000	0.966	0.993	0.995	0.661	0.681	0.664
0.95	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.767	0.765	0.782
0.99	1 000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.881	0.894	0.894
1.00	1.000	1.000	1.000	1.000	1.00C	1.000	1.000	1.000	1.000	0.937	0.952	0.938

Bandwidth = $\ell(4)$

ρ	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.00	0.053	0.055	0.050	0.057	0.055	0.045	0.057	0.052	0.041	0.049	0.059	0.055
0.10	0.051	0.051	0.052	0.047	0.049	0.038	0.059	0.050	0.042	0.056	0.056	0.049
0.50	0.082	0.093	0.082	0.086	0.099	0.084	0.082	0.090	0.085	0.109	0.069	0.082
0.80	0.184	0.252	0.205	0.198	0.250	0.223	0.189	0.228	0.225	0.153	0.147	0.146
0.85	0.266	0.375	0.333	0.285	0.380	0.345	0.263	0.338	0.342	0.210	0.219	0.202
0.90	0.371	0.514	0.432	0.408	0.528	0.491	0.370	0.465	0.454	0.270	0.263	0.273
0.95	0.628	0.793	0.722	0.695	0.799	0.752	0.614	0.740	0.728	0.381	0.411	0.430
0.99	0.959	0.989	0.971	0.974	0.992	0.975	0.950	0.977	0.977	0.652	0.798	0.702
1.00	0.994	0.999	0.997	0.997	0.999	0.997	0.993	0.997	0.996	0.839	0.864	0.835

Bandwidth = $\ell(12)$

ρ	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.00	0.039	0.041	0.033	0.040	0.037	0.032	0.043	0.031	0.035	0.041	0.055	0.043
0.10	0.057	0.052	0.043	0.057	0.048	0.045	0.055	0.050	0.040	0.056	0.061	0.063
0.50	0.049	0.051	0.041	0.050	0.046	0.045	0.052	0.048	0.034	0.045	0.057	0.053
0.80	0.081	0.085	0.068	0.073	0.066	0.043	0.080	0.083	0.059	0.075	0.066	0.070
0.85	0.101	0.108	0.102	0.097	0.101	0.065	0.104	0.102	0.080	0.085	0.105	0.084
0.90	0.144	0.172	0.126	0.144	0.163	0.106	0.144	0.166	0.133	0.123	0.143	0.131
0.95	0.260	0.318	0.270	0.265	0.312	0.236	0.261	0.306	0.265	0.190	0.221	0.220
0.99	0.727	0.806	0.722	0.754	0.803	0.701	0.739	0.800	0.746	0.461	0.527	0.505
1.00	0.939	0.974	0.931	0.958	0.974	0.936	0.939	0.966	0.947	0.726	0.780	0.729

Table 3.6: Power: T=200. Bandwidth=$f(0)$

Uniformly Distributed Structural Break

BrkSize	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.069	0.062	0.064	0.068	0.059	0.043	0.070	0.066	0.049	0.062	0.071	0.073
0.2	0.150	0.146	0.133	0.147	0.143	0.100	0.154	0.146	0.116	0.115	0.140	0.097
0.5	0.562	0.618	0.571	0.575	0.607	0.535	0.572	0.606	0.573	0.402	0.454	0.362
1.0	0.836	0.889	0.830	0.865	0.909	0.910	0.827	0.848	0.845	0.660	0.728	0.675
2.0	0.914	0.962	0.919	0.940	0.969	0.972	0.905	0.920	0.919	0.814	0.855	0.812
5.0	0.971	0.996	0.972	0.980	0.990	0.993	0.965	0.972	0.970	0.910	0.937	0.894

Unit Root

σ	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.01	0.099	0.100	0.093	0.093	0.094	0.062	0.102	0.099	0.075	0.075	0.091	0.074
0.02	0.259	0.265	0.238	0.254	0.249	0.194	0.261	0.259	0.217	0.171	0.222	0.178
0.05	0.607	0.634	0.617	0.619	0.630	0.572	0.616	0.628	0.598	0.481	0.520	0.461
0.10	0.848	0.878	0.853	0.865	0.882	0.857	0.843	0.862	0.861	0.684	0.705	0.653
0.20	0.961	0.974	0.960	0.968	0.978	0.973	0.957	0.968	0.965	0.793	0.835	0.795
0.50	0.996	0.997	0.996	0.997	0.998	0.998	0.994	0.995	0.996	0.852	0.877	0.855

Fractional Integration

d	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.25	0.579	0.687	0.651	0.612	0.705	0.668	0.587	0.647	0.654	0.389	0.416	0.388
0.50	0.939	0.983	0.967	0.961	0.983	0.981	0.935	0.965	0.964	0.703	0.721	0.657
0.75	0.995	0.998	0.997	0.997	0.999	0.999	0.992	0.997	0.997	0.811	0.854	0.830
1.00	1.000	1.000	1.000	1.000	1.000	1.000	0.998	1.000	1.000	0.898	0.907	0.881

Structural Break

BrkPt	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.169	0.243	0.125	0.223	0.279	0.252	0.161	0.177	0.166	0.203	0.098	0.055
0.2	0.520	0.617	0.501	0.565	0.609	0.515	0.552	0.624	0.591	0.700	0.300	0.128
0.3	0.746	0.774	0.746	0.739	0.751	0.662	0.759	0.777	0.736	0.799	0.541	0.208
0.4	0.842	0.842	0.836	0.821	0.816	0.729	0.839	0.837	0.795	0.646	0.782	0.377
0.5	0.870	0.868	0.869	0.847	0.838	0.753	0.867	0.864	0.808	0.479	0.944	0.509
0.6	0.858	0.860	0.864	0.842	0.837	0.763	0.862	0.862	0.827	0.373	0.824	0.670
0.7	0.725	0.758	0.727	0.727	0.737	0.663	0.741	0.762	0.721	0.217	0.540	0.796
0.8	0.529	0.594	0.480	0.560	0.587	0.510	0.543	0.611	0.586	0.126	0.285	0.659
0.9	0.181	0.267	0.140	0.240	0.327	0.299	0.173	0.201	0.198	0.066	0.098	0.219

Table 3.7: Power: T=200. Bandwidth=$\ell(4)$

Uniformly Distributed Structural Break

BrkSize	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.064	0.058	0.054	0.062	0.055	0.032	0.064	0.056	0.042	0.056	0.068	0.060
0.2	0.118	0.128	0.098	0.123	0.113	0.074	0.123	0.122	0.098	0.115	0.109	0.103
0.5	0.534	0.563	0.521	0.541	0.541	0.444	0.540	0.559	0.507	0.336	0.439	0.349
1.0	0.795	0.872	0.788	0.840	0.882	0.891	0.781	0.825	0.831	0.587	0.644	0.571
2.0	0.897	0.953	0.880	0.925	0.965	0.970	0.878	0.898	0.903	0.716	0.778	0.712
5.0	0.933	0.994	0.914	0.973	0.989	0.989	0.917	0.933	0.931	0.742	0.822	0.755

Unit Root

σ	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.01	0.077	0.073	0.054	0.071	0.064	0.049	0.082	0.075	0.053	0.064	0.081	0.078
0.02	0.238	0.239	0.212	0.242	0.225	0.159	0.245	0.242	0.193	0.154	0.212	0.164
0.05	0.568	0.590	0.538	0.577	0.592	0.503	0.566	0.580	0.551	0.405	0.480	0.412
0.10	0.771	0.808	0.768	0.787	0.805	0.753	0.774	0.799	0.782	0.577	0.656	0.603
0.20	0.886	0.919	0.877	0.906	0.924	0.890	0.886	0.913	0.900	0.719	0.754	0.680
0.50	0.945	0.973	0.929	0.954	0.973	0.940	0.947	0.966	0.949	0.723	0.777	0.725

Fractional Integration

d	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.25	0.302	0.353	0.309	0.308	0.348	0.271	0.313	0.348	0.313	0.242	0.267	0.218
0.50	0.652	0.737	0.650	0.696	0.740	0.633	0.666	0.726	0.675	0.444	0.518	0.464
0.75	0.848	0.907	0.845	0.871	0.908	0.854	0.845	0.896	0.863	0.608	0.666	0.643
1.00	0.946	0.970	0.934	0.960	0.972	0.929	0.943	0.965	0.944	0.726	0.773	0.724

Structural Break

BrkPt	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.146	0.216	0.093	0.193	0.248	0.204	0.135	0.152	0.143	0.198	0.086	0.046
0.2	0.485	0.548	0.412	0.521	0.537	0.426	0.503	0.566	0.528	0.649	0.256	0.086
0.3	0.728	0.747	0.718	0.720	0.725	0.582	0.745	0.765	0.681	0.792	0.500	0.177
0.4	0.816	0.810	0.802	0.776	0.772	0.657	0.817	0.806	0.735	0.621	0.747	0.276
0.5	0.849	0.834	0.842	0.817	0.798	0.668	0.849	0.832	0.761	0.426	0.935	0.427
0.6	0.841	0.829	0.837	0.810	0.805	0.669	0.839	0.825	0.765	0.243	0.767	0.643
0.7	0.700	0.737	0.683	0.692	0.698	0.580	0.720	0.750	0.669	0.160	0.477	0.786
0.8	0.448	0.513	0.391	0.486	0.510	0.410	0.470	0.530	0.508	0.078	0.210	0.642
0.9	0.157	0.211	0.095	0.195	0.244	0.225	0.144	0.154	0.133	0.055	0.089	0.227

Table 3.8: Power: T=200. Bandwidth=$\ell(12)$

Uniformly Distributed Structural Break

BrkSize	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.068	0.057	0.043	0.062	0.051	0.031	0.072	0.058	0.030	0.069	0.071	0.065
0.2	0.127	0.101	0.075	0.124	0.088	0.040	0.132	0.101	0.055	0.093	0.122	0.093
0.5	0.430	0.425	0.354	0.423	0.406	0.176	0.445	0.432	0.299	0.243	0.352	0.266
1.0	0.713	0.817	0.651	0.754	0.856	0.791	0.713	0.760	0.762	0.439	0.524	0.413
2.0	0.800	0.932	0.715	0.872	0.959	0.961	0.785	0.816	0.823	0.474	0.585	0.473
5.0	0.843	0.991	0.756	0.951	0.986	0.988	0.808	0.843	0.848	0.502	0.603	0.502

Unit Root

σ	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.01	0.093	0.075	0.063	0.086	0.064	0.038	0.091	0.078	0.039	0.064	0.095	0.080
0.02	0.165	0.150	0.121	0.157	0.127	0.048	0.175	0.154	0.091	0.123	0.157	0.126
0.05	0.470	0.462	0.381	0.466	0.429	0.206	0.479	0.463	0.325	0.324	0.410	0.310
0.10	0.615	0.631	0.541	0.611	0.618	0.384	0.624	0.636	0.526	0.445	0.545	0.425
0.20	0.696	0.714	0.616	0.691	0.704	0.475	0.704	0.716	0.606	0.497	0.603	0.496
0.50	0.720	0.752	0.647	0.726	0.735	0.485	0.730	0.753	0.640	0.531	0.635	0.542

Fractional Integration

d	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.25	0.158	0.161	0.099	0.148	0.131	0.037	0.170	0.164	0.073	0.135	0.148	0.135
0.50	0.361	0.355	0.255	0.343	0.331	0.121	0.366	0.363	0.217	0.259	0.333	0.255
0.75	0.574	0.605	0.468	0.580	0.586	0.286	0.579	0.607	0.439	0.403	0.493	0.393
1.00	0.707	0.752	0.599	0.722	0.739	0.457	0.711	0.749	0.628	0.526	0.603	0.524

Structural Break

BrkPt	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.107	0.116	0.049	0.136	0.155	0.117	0.108	0.095	0.066	0.153	0.074	0.045
0.2	0.324	0.370	0.158	0.367	0.366	0.163	0.354	0.413	0.312	0.594	0.156	0.050
0.3	0.588	0.601	0.492	0.555	0.533	0.201	0.623	0.622	0.436	0.713	0.357	0.069
0.4	0.714	0.697	0.681	0.668	0.613	0.230	0.717	0.699	0.487	0.466	0.657	0.144
0.5	0.781	0.746	0.720	0.716	0.637	0.252	0.779	0.742	0.484	0.288	0.917	0.279
0.6	0.708	0.686	0.669	0.645	0.590	0.247	0.720	0.688	0.479	0.160	0.671	0.459
0.7	0.570	0.584	0.448	0.558	0.523	0.200	0.609	0.603	0.402	0.094	0.350	0.703
0.8	0.339	0.400	0.183	0.371	0.385	0.179	0.373	0.442	0.335	0.054	0.175	0.614
0.9	0.104	0.124	0.038	0.136	0.169	0.134	0.106	0.091	0.060	0.032	0.072	0.184

Table 3.9: Power: T=800. Bandwidth=$f(0)$

Uniformly Distributed Structural Break

BrkSize	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.145	0.148	0.134	0.137	0.147	0.124	0.144	0.138	0.125	0.109	0.120	0.107
0.2	0.423	0.443	0.419	0.431	0.429	0.391	0.428	0.444	0.427	0.296	0.369	0.282
0.5	0.839	0.898	0.849	0.872	0.908	0.917	0.822	0.858	0.862	0.668	0.686	0.674
1.0	0.915	0.953	0.916	0.936	0.955	0.960	0.901	0.915	0.916	0.820	0.850	0.807
2.0	0.957	0.982	0.954	0.969	0.976	0.978	0.950	0.952	0.954	0.908	0.924	0.901
5.0	0.984	0.998	0.984	0.989	0.994	0.997	0.982	0.983	0.985	0.960	0.973	0.963

Unit Root

σ	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.01	0.498	0.519	0.496	0.506	0.513	0.473	0.499	0.514	0.500	0.386	0.420	0.391
0.02	0.794	0.833	0.807	0.808	0.834	0.824	0.792	0.818	0.814	0.643	0.696	0.623
0.05	0.976	0.987	0.985	0.979	0.986	0.985	0.973	0.983	0.986	0.827	0.854	0.826
0.10	0.997	1.000	0.999	0.998	1.000	1.000	0.996	0.999	1.000	0.896	0.897	0.900
0.20	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.930	0.934	0.924
0.50	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.939	0.939	0.919

Fractional Integration

d	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.25	0.795	0.914	0.880	0.841	0.923	0.938	0.783	0.868	0.887	0.519	0.532	0.550
0.50	0.999	1.000	1.000	0.999	1.000	1.000	0.998	0.998	0.999	0.823	0.851	0.819
0.75	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.898	0.914	0.918
1.00	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.949	0.957	0.952

Structural Break

BrkPt	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.649	0.884	0.704	0.814	0.911	0.920	0.579	0.732	0.772	0.695	0.310	0.147
0.2	0.990	0.999	0.996	0.993	0.999	0.996	0.991	0.999	0.999	0.997	0.794	0.345
0.3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.987	0.659
0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997	1.000	0.892
0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.987	1.000	0.968
0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.896	1.000	0.999
0.7	1.000	1.000	1.000	0.999	1.000	0.999	1.000	1.000	1.000	0.661	0.991	1.000
0.8	0.992	0.999	0.998	0.997	0.999	0.998	0.994	0.999	0.999	0.358	0.794	0.999
0.9	0.605	0.893	0.665	0.783	0.910	0.925	0.548	0.692	0.737	0.137	0.270	0.651

Table 3.10: Power: T=800, Bandwidth=$\ell(4)$

Uniformly Distributed Structural Break

BrkSize	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.133	0.127	0.130	0.130	0.122	0.086	0.135	0.124	0.104	0.111	0.117	0.099
0.2	0.421	0.443	0.407	0.420	0.430	0.358	0.425	0.439	0.405	0.293	0.336	0.288
0.5	0.828	0.888	0.830	0.859	0.896	0.903	0.815	0.848	0.852	0.635	0.701	0.668
1.0	0.919	0.967	0.918	0.940	0.970	0.974	0.901	0.919	0.920	0.798	0.843	0.790
2.0	0.954	0.984	0.941	0.965	0.979	0.982	0.941	0.946	0.948	0.855	0.891	0.867
5.0	0.978	1.000	0.970	0.988	0.994	0.996	0.971	0.976	0.978	0.913	0.942	0.905

Unit Root

σ	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.01	0.510	0.532	0.508	0.519	0.530	0.475	0.512	0.528	0.515	0.367	0.456	0.378
0.02	0.747	0.779	0.764	0.760	0.780	0.752	0.752	0.770	0.761	0.591	0.652	0.605
0.05	0.952	0.973	0.956	0.962	0.974	0.972	0.946	0.961	0.961	0.752	0.816	0.785
0.10	0.989	0.994	0.992	0.993	0.994	0.994	0.983	0.992	0.992	0.849	0.851	0.843
0.20	0.997	0.998	0.997	0.997	0.998	0.998	0.996	0.998	0.998	0.837	0.866	0.826
0.50	0.997	1.000	0.999	0.999	1.000	0.999	0.996	0.998	0.998	0.851	0.876	0.847

Fractional Integration

d	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.25	0.501	0.589	0.545	0.524	0.589	0.559	0.497	0.563	0.560	0.347	0.380	0.353
0.50	0.877	0.935	0.904	0.905	0.933	0.925	0.873	0.918	0.919	0.644	0.659	0.647
0.75	0.974	0.990	0.978	0.984	0.989	0.986	0.969	0.980	0.978	0.740	0.781	0.765
1.00	0.998	1.000	0.999	1.000	1.000	1.000	0.996	0.998	0.998	0.856	0.878	0.870

Structural Break

BrkPt	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.591	0.866	0.619	0.751	0.905	0.919	0.523	0.669	0.718	0.654	0.234	0.090
0.2	0.988	0.996	0.994	0.991	0.996	0.996	0.989	0.996	0.996	0.996	0.736	0.295
0.3	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	0.999	1.000	0.977	0.595
0.4	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	0.999	1.000	0.841
0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.956	1.000	0.957
0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.853	1.000	0.996
0.7	0.999	0.999	1.000	0.999	0.999	0.999	0.999	0.999	1.000	0.581	0.981	1.000
0.8	0.984	0.999	0.997	0.988	1.000	0.998	0.987	1.000	0.998	0.274	0.732	0.998
0.9	0.559	0.850	0.568	0.736	0.897	0.905	0.492	0.633	0.683	0.089	0.242	0.622

Table 3.11: Power: T=800, Bandwidth=ℓ(12)

Uniformly Distributed Structural Break

BrkSize	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.133	0.125	0.120	0.124	0.120	0.095	0.135	0.122	0.107	0.091	0.129	0.098
0.2	0.389	0.395	0.377	0.378	0.379	0.299	0.390	0.395	0.350	0.251	0.322	0.256
0.5	0.779	0.838	0.779	0.807	0.856	0.869	0.778	0.796	0.804	0.567	0.653	0.555
1.0	0.876	0.950	0.872	0.904	0.965	0.973	0.860	0.880	0.887	0.678	0.761	0.677
2.0	0.927	0.985	0.906	0.958	0.983	0.986	0.911	0.923	0.927	0.751	0.818	0.721
5.0	0.952	0.999	0.928	0.980	0.994	0.996	0.929	0.943	0.945	0.747	0.841	0.760

Unit Root

σ	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.01	0.458	0.473	0.444	0.460	0.462	0.401	0.467	0.467	0.446	0.347	0.401	0.341
0.02	0.710	0.758	0.729	0.728	0.756	0.698	0.712	0.746	0.735	0.563	0.613	0.556
0.05	0.878	0.919	0.885	0.893	0.917	0.882	0.883	0.914	0.890	0.685	0.759	0.683
0.10	0.942	0.975	0.938	0.957	0.978	0.942	0.936	0.965	0.951	0.713	0.789	0.717
0.20	0.934	0.965	0.917	0.953	0.969	0.939	0.933	0.959	0.944	0.732	0.770	0.741
0.50	0.953	0.976	0.946	0.962	0.977	0.944	0.954	0.969	0.955	0.779	0.818	0.733

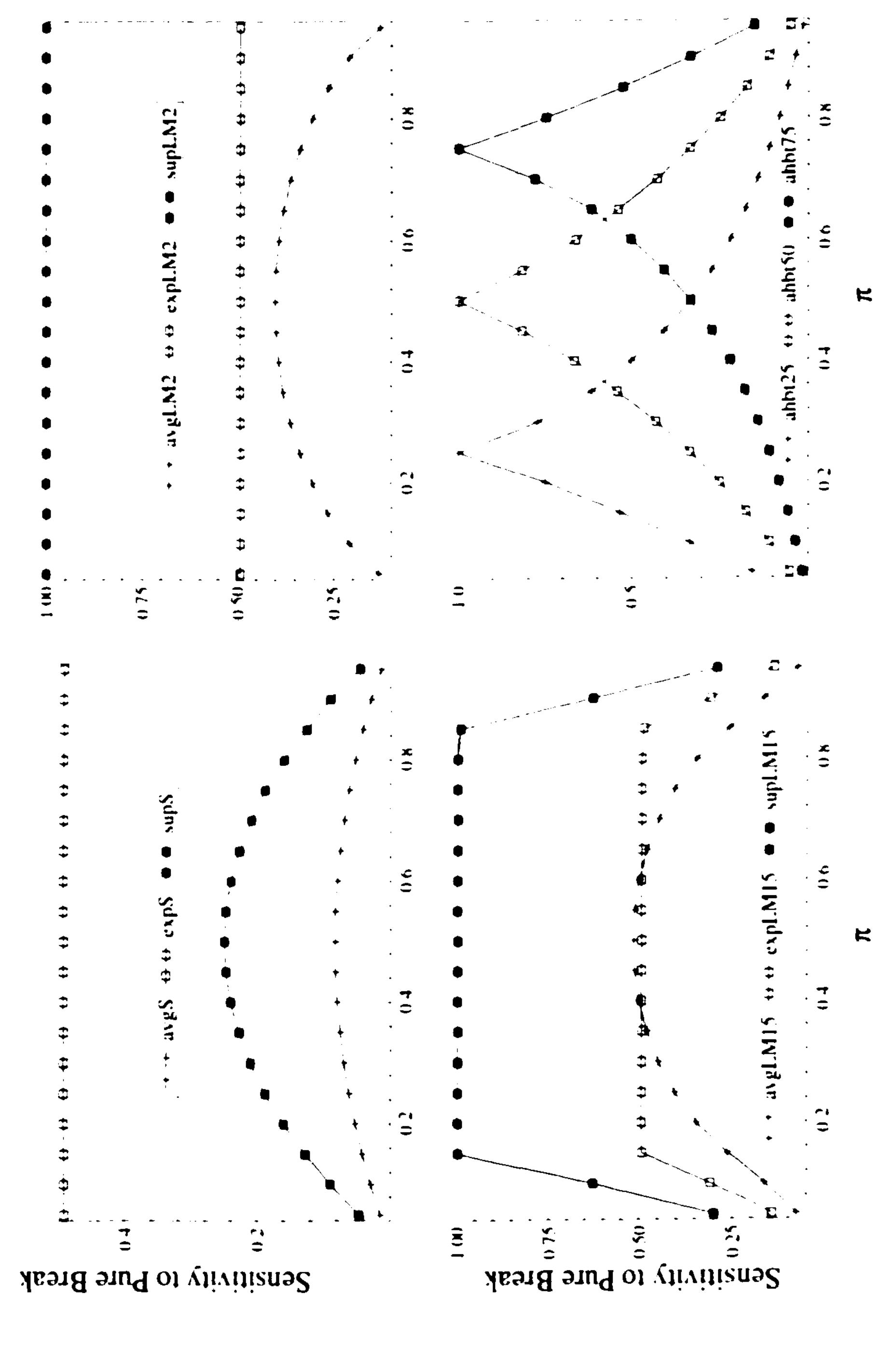
Fractional Integration

d	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.25	0.293	0.349	0.307	0.308	0.341	0.265	0.300	0.339	0.308	0.216	0.260	0.228
0.50	0.688	0.758	0.670	0.717	0.751	0.675	0.687	0.744	0.703	0.473	0.546	0.488
0.75	0.860	0.913	0.854	0.878	0.908	0.831	0.859	0.901	0.873	0.631	0.700	0.623
1.00	0.919	0.953	0.899	0.942	0.957	0.914	0.913	0.940	0.921	0.700	0.757	0.720

Structural Break

BrkPt	avgS	expS	supS	aLM_2	eLM_2	sLM_2	aLM_{15}	eLM_{15}	sLM_{15}	ah_{25}	ah_{50}	ah_{75}
0.1	0.439	0.827	0.387	0.645	0.871	0.882	0.384	0.523	0.591	0.577	0.171	0.055
0.2	0.963	0.996	0.989	0.979	0.996	0.995	0.972	0.995	0.999	0.998	0.608	0.158
0.3	0.998	0.999	1.000	0.999	0.999	0.999	0.998	0.999	1.000	1.000	0.942	0.353
0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.988	0.999	0.643
0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.893	1.000	0.892
0.6	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	0.671	0.999	0.987
0.7	0.999	1.000	1.000	0.999	1.000	0.998	0.999	1.000	1.000	0.374	0.956	1.000
0.8	0.961	0.993	0.981	0.976	0.992	0.989	0.970	0.994	0.994	0.165	0.624	0.997
0.9	0.436	0.779	0.380	0.623	0.854	0.871	0.372	0.518	0.566	0.055	0.173	0.569

Figure 3.1: Break Sensitivity Functions



Chapter 4

Extreme Values and Local Power for KPSS Tests

We establish upper bounds on the statistics $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ proposed by Kwiatkowski, Phillips, Schmidt, and Shin (1992) for testing the null hypothesis of stationarity. The bounds are attained by cosine functions that do not exhibit the exploding variance often associated with the alternative hypothesis. We also derive extreme value results for time trends, structural breaks, and unit roots.

Using these results for extreme KPSS statistics, we develop analytic results for the power against local alternatives that combine one of these processes with a short memory component. The results are surprising, given the origins of the statistic $\hat{\eta}_\mu$, in that the rejection probabilities for the test based on $\hat{\eta}_\mu$ are very similar for structural breaks and unit roots. We provide a theoretical basis for this last result by showing that $\hat{\eta}_\mu$ is an algebraic special case of the statistic proposed by Andrews and Ploberger (1994) for testing for structural breaks at an unknown breakpoint.

4.1 Introduction

We establish upper bounds for the statistics $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ proposed by Kwiatkowski, Phillips, Schmidt, and Shin (1992). While these bounds are of some interest in their own right because they define an extreme opposite to the null hypothesis, they also provide a basis for analyzing the power of tests based on $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$. In particular, our results for extreme values place an analytic perspective on power comparisons for unit root, time trend, and structural break alternatives. We demonstrate that $\hat{\eta}_\mu$ is about as effective as a test for structural breaks as it is as a test for unit roots.

Our results add to earlier findings that have already revised the interpretation of these statistics. Kwiatkowski, Phillips, Schmidt, and Shin (KPSS) derive $\hat{\eta}_\mu$ as an LM statistic where the alternative hypothesis is the sum of a unit root process and a stationary component. Their stated null hypothesis is a short-memory, stationary process. On that basis, they describe the value of $\hat{\eta}_\mu$ as a test of the null hypothesis of stationarity. Lee and Schmidt (1996) and Liu (1998) show that $\hat{\eta}_\mu$ is consistent for fractionally integrated I(d) processes with $d > 0$. This result necessarily revises the interpretation of the KPSS test because an I(d) process with $0 < d < 1/2$ is stationarity. Lee and Schmidt propose that the KPSS test is more properly viewed as a test of the null hypothesis of short memory. The alternative hypothesis would then be long memory, which includes unit roots and fractionally integrated processes.

Unit roots and fractionally integrated processes do not, however, yield the largest values for $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$. For a sample of size T, we establish the bounds

$$\hat{\eta}_\mu \leq \pi^{-2}T$$

and

$$\hat{\eta}_\tau \leq (2\pi)^{-2}T.$$

These bounds are attained for the processes $y_t = cos(\pi t/T)$ and $y_t = cos(2\pi t/T)$, respectively. These cosine functions provide benchmarks for other possible alternative hypotheses, and they give a clear picture of the types of realizations that actually trigger large values for $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$. They show that $\hat{\eta}_\mu$ is most sensitive to a large change in the value of y_t between the beginning and the end of the sample and that $\hat{\eta}_\tau$ is most sensitive to data with a complete cycle in the data period.

The results for time trends and structural breaks confirm this observation. The value $\hat{\eta}_\mu = \frac{1}{10}T$ is attained by a simple time trend, $y_t = \alpha t$, where α is a constant. The difference between $\frac{1}{10}T$ for a time trend and $\pi^{-2}T = 0.1013T$ for a cosine function is minimal. A structural break process with a break at the sample midpoint is a crude approximation to a monotone trend, and that process attains $\hat{\eta}_\mu = \frac{1}{12}T$.

Unit roots, which form the original basis for the statistics, generate realizations that may or may not resemble cosine functions. The values of $\hat{\eta}_\mu$ for unit root realizations are more or less uniformly distributed between the lower bound of 0 and the upper bound of $\pi^{-2}T$. The mean is at the midpoint of $\frac{1}{20}T$, which is about half of the bound $\pi^{-2}T = 0.1013T$. Only about 12% of realized unit roots have a value of $\hat{\eta}_\mu$ as large as the value $\frac{1}{12}T$ for a structural break known to be at the sample midpoint. The mean of $\frac{1}{18}T$ for a break with an unknown, uniformly distributed breakpoint is also larger than the mean for a unit root.

These extreme value results establish a framework for studying the power of the KPSS tests for alternatives that include unit roots and structural breaks. We consider an alternative hypothesis based on the process

$$z_t = x_t + \gamma_T y_t$$

composed of the sum of a short memory process x_t and a long memory process y_t. The scalar γ_T sets the distance between the null and the alternative for a sample of size T. We show that the power for a small γ_T is closely related to the extreme value results for the process y_t alone.

Section 4.2 of this paper establishes the bounds for $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$. Section 4.3 presents a result on asymptotic local power for processes that combine short memory and long memory components. Section 4.4 then presents some calculations for various unit roots, time trends, and structural breaks. Simulation results in Section 4.5 give detailed power results for various alternatives.

We provide a theoretical basis for our finding that $\hat{\eta}_\mu$ is about as effective in testing for structural breaks as it is in testing for unit roots by showing that it is an algebraic special case of the test Andrews and Ploberger (1994) propose for structural breaks. This derivation is given in Section 4.6, and Section 4.7 presents our summary and conclusions.

4.2 Bounds on KPSS Statistics

The KPSS statistics $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ are computed from two types of residuals e_t calculated for a process y_t. The statistic $\hat{\eta}_\mu$ is based on the residuals from the regression of y_t on an intercept. The statistic $\hat{\eta}_\tau$ is similarly calculated by regressing y_t on a constant term and a time trend.

In both cases, the fundamental quantity is the partial sum of residuals

$$S_t = \sum_{t=1}^{t} e_t. \tag{4.1}$$

If the residuals are short memory, the long-run variance

$$\sigma^2 = \lim_{T\to\infty} T^{-1}E(S_T^2) \tag{4.2}$$

is finite. If not, the residuals e_t are said to have the long memory property.

The KPSS statistics can be written as

$$\hat{\eta} = \frac{T^{-2}\sum_{t=1}^{T} S_t^2}{s^2(\ell)}. \tag{4.3}$$

for $\hat{\eta}_\mu$ or $\hat{\eta}_\tau$. The denominator is a kernel-based estimate of the long-run variance σ^2:

$$s^2(\ell) = T^{-1}\sum_{t=1}^{T} e_t^2 + 2T^{-1}\sum_{s=1}^{\ell} w(s,\ell) \sum_{t=s+1}^{T} e_t e_{t-s}. \tag{4.4}$$

where $w(s,\ell) = 1 - s/(\ell+1)$ is the Bartlett window with bandwidth ℓ. For testing under the null hypothesis of short memory, ℓ is chosen so that $\ell = o_p(T^{1/2})$ to obtain a consistent estimate of σ^2.

For the present study of extreme values for $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ we choose $\ell = 0$ even though σ^2 does not exist and cannot be consistently estimated by any choice of the bandwidth. We are interested here in extreme values for $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$, which occur for process that are not short memory. The sample variance $s^2(0)$ does, nonetheless, have a finite expectation in a finite sample, and its value provides a convenient reference point that we use in the next section to examine the powers of the statistics.

We also simplify the analysis by studying continuous time limits. For time trends and structural breaks, the continuous time version is immediate and we will use the notation $e_t \to e(t)$ to denote convergence to that limit. For stochastic process, the notation $e_t \to e(t)$ will denote weak convergence in measure to a demeaned Brownian motion for a unit root or to demeaned fractional Brownian motion for a fractionally integrated process.

The fundamental problem we address in this section is thus maximizing the functional

$$\tilde{\eta}^* = \frac{\int_0^1 \left(\int_0^a e(s)ds\right)^2 da}{\int_0^1 e(s)^2 ds}. \tag{4.5}$$

over $e(s)$, $s \in [0, 1]$. While the limits $e(t)$ for stochastic processes are not smooth, we will assume that the maximum for $\hat{\eta}^*$ is attained for a smooth, differentiable process. We let $\hat{\eta}^*_\mu$ denote the maximum subject to the constraint

$$\int_0^1 e(s)ds = 0 \tag{4.6}$$

We let $\hat{\eta}^*_\tau$ denote the maximum subject to (4.6) and the additional constraint

$$\int_0^1 s \cdot e(s)\, ds = 0 \tag{4.7}$$

Proposition 2. *For bandwidth $\ell = 0$, the limiting KPSS statistics defined above satisfy the following bounds:*

$$\hat{\eta}^*_\mu \leq \pi^{-2}T$$

$$\hat{\eta}^*_\tau \leq (2\pi)^{-2}T$$

Proof. See Section 4.8.1. □

The bound $\hat{\eta}^*_\mu = \pi^{-2}T$ occurs for the data $y_t = y_0 \cos(\pi t/T)$, which is a cosine wave with a period of $2T$. This function has a peak at the beginning of the sample, and a trough at the end. (The function $y_t = y_0 \cos(\pi(1 - t/T))$ is a mirror image that also attains the maximum.) The largest possible detrended KPSS statistic $\hat{\eta}_\tau$ occurs for the data $y_t = y_0 \cos(2\pi t/T)$. This cosine function has a period equal to the sample size with peaks at the beginning and end of the sample.

4.3 Local Power

The bounds on $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ have implications for the more practical problem of detecting a long memory component in a process that also contains a substantial short memory component. Consider a process

$$z_t = x_t + \gamma_T y_t \tag{4.8}$$

composed of the sum of a short memory process x_t and a long memory process y_t. The scalar γ_T sets the extent to which the alternative hypothesis differs from the null. We assume that

$\gamma_T^2 = O_p(T^{-1})$. Let $\hat{\eta}_x$ and $\hat{\eta}_z$ be the KPSS statistics for x_t and z_t calculated using bandwidth ℓ. (For notational clarity, we will generally drop the subscripts μ and τ. We note that the algebra for the two cases is the same.)

The critical values for the test are based on the distribution of $\hat{\eta}_x$. If we let $g(\eta)$ denote the density for the KPSS statistic, then

$$Pr(\hat{\eta}_x > c) = \int_c^\infty g(\eta)d\eta$$

We let c^* denote a particular critical value with $Pr(\hat{\eta}_x > c^*) = \alpha^*$.

We will show that the local power of the test depends on y_t only through the expectation of

$$\Psi = T^{-2}\sum_{t=1}^{T} S_{y\,t}^2. \tag{4.9}$$

While $E(\Psi)$ generally grows without bound, it does exist for a finite T. We define

$$v_T = \frac{E(\Psi)}{T}$$

and assume that the parameters of the process y_t are chosen so that $v_T \equiv V_y$ for a constant V_y. In the cases we consider, $s_y^2(0)$ will be constant when $v_T \equiv V_y$.

Proposition 3. *If $\gamma_T^2 = T^{-1}$, then the rejection probabilities under the alternative satisfy*

$$Pr(\hat{\eta}_z > c^*) - \alpha^* \to g(c^*)\frac{V_y}{\sigma_x^2} \quad as \quad T \to \infty. \tag{4.10}$$

Proof. See Section 4.8.2. □

The asymptotic local power of the KPSS test depends on the alternative only through V_y/σ_x^2 and will thus be the same for all alternatives with a given value for V_y. Proposition 3 establishes a square root rule: quadrupling T and halving γ_T yields constant rejection probabilities.

Proposition 2 can be used to establish a lower bound on the sample variance $E(s_y^2(0))$ necessary to achieve a given value for V_y. In the case of $\hat{\eta}_\mu$, for example, the bound in Proposition 2 implies that

$$s_y^2(0) \geq \pi^2 T^{-3}\sum_{t=1}^{T} S_{y\,t}^2.$$

Taking expectations of both sides yields

$$E(s_y^2(0)) \geq \pi^2 V_y \tag{4.11}$$

The cosine function that establishes the bound for $\hat{\eta}_\mu$ also has the minimum possible variance necessary to achieve a given value for V_y. In the next section, we address the practical question of how large $E(s_y^2(0))$ must be to achieve a given V_y for processes that do not attain the bound in Proposition 2.

4.4 Alternative Hypotheses

Cosine functions are certainly not the most frequently mentioned alternative hypotheses for the KPSS tests. That honor accrues to the unit root process, with fractional integration a distant second. We augment this range with a variety of structural break processes and time trends.

Table 4.1 specifies a variety of process that have large values for $\hat{\eta}_\mu$. The parameters of each process are chosen to achieve $V_y = 1$ so that all the processes have the same asymptotic local power as given in Proposition 3. Table 4.1 presents $E(s_y^2(0))$ and $E(\hat{\eta}_\mu)$ for these processes. These results are derived in Section 4.8.3.

The unit root process that motivated the original derivation of $\hat{\eta}_\mu$ does not dominate the set of extreme values for the KPSS statistics. The expected value of $\hat{\eta}_\mu$ for a pure unit root (process UR) is very close to $\frac{1}{20}T$, which is half the bound set by the cosine function. Table 4.3 shows the distribution under a unit root, which is surprisingly uniform between 0 and $\pi^{-2}T$. This distribution reflects an attribute of unit roots; some unit root realizations are clearly nonstationary, but others are by chance not all that distinct in their appearance from stationary data.

Indeed, $\hat{\eta}_\mu$ is more sensitive to a variety of processes than it is to the process denoted UR. The value $\hat{\eta}_\mu = \frac{1}{10}T$ for a simple time trend (process TT) is a close second to the value for a cosine function. The difference between $\frac{1}{10}$ and $\pi^{-2} = 0.10132$ is just a little over 1%. The value $\hat{\eta}_\mu = \frac{1}{12}T$ for a simple break at the sample midpoint (process $D_{1/2}$) is greater than $\hat{\eta}_\mu$ for 88% of realized unit roots. If the break is at the point $[\theta T]$, where $[\cdot]$ denotes the integer closest to a real number, then $\hat{\eta}_\mu = \frac{1}{3}\theta(1-\theta)T$. The average for a break uniformly distributed in $[0,1]$ is $\frac{1}{18}T$.

These results have parallel implications for asymptotic local power. The sample variances needed to achieve $V_y = 1$ are 9.87, 10, and 12 for COS, TT, and $D_{1/2}$, but 15 for UR. The latter figure is slightly smaller that the sample variance of 16 needed to attain $V_y = 1$ for the unknown breakpoint process (process D_θ).

Table 4.2 also shows corresponding results incorporating a time trend. Again, the cosine sets the bound on $\hat{\eta}_\tau$ and has the smallest sample variance that achieves $V_y = 1$. A close second place goes to the broken trend that goes up linearly over the first half of the sample and down linearly over the second half. The statistic $\hat{\eta}_\tau$ equals $\frac{1}{40}T$ for the latter case. This value of $0.025T$ is only slightly less than $(2\pi)^{-2}T = 0.025337T$. The double break process that takes on one value between 1/2 and 3/4 and another value on the other two regions has $\hat{\eta}_\tau = \frac{1}{48}T$ or $0.0208T$.

Table 4.3 shows the distribution of $\hat{\eta}_\tau$ for a pure unit root. The distribution covers the range between 0 and $(2\pi)^{-2}T$, but it is not as uniform as the distribution of $\hat{\eta}_\mu$. The mean is about $0.0109T$.

4.5 Power Comparisons

The powers of $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ for the various alternatives described above are partially dictated by Proposition 2 and Proposition 3. Proposition 2 implies that the power will be bounded from above by the powers for the cosine functions. Proposition 3 gives an asymptotic local power approximation, showing the local powers are equal for processes with equal values for V_y. The remaining issue is the very practical question of power for larger values of γ for processes that do not attain the bound in Proposition 2. We approach this issue using simulations.

Tables 4.4 and 4.5 present rejection probabilities for a variety of alternative hypotheses for $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$, respectively. In all cases, the short memory component is $x_t \sim N(0,1)$. The alternatives y_t are described in Tables 4.1 and 4.2. The alternatives are parameterized so that $V_y = 1$, equating asymptotic local powers, and that characteristic is evident in the results for small γ_t. The results also confirm that the cosine functions bound the powers. The results (after scaling) are very nearly identical across sample sizes of 50, 200, and 800.

The results for the three non-stochastic processes (COS, TT, and $D_{1/2}$ in Table 4.4 and

(COS2, TT2, and D2 in Table 4.5) are very nearly identical for all values of γ_T. The precise shape of the y_t process is evidently less important than the fact that the shape is non-stochastic. The power for unit roots with the same V_y is similar for small values of γ_T but lower for larger values of γ_T.

To put the power for unit roots into perspective, we compare those figures with powers for four types of structural break processes in Table 4.4.[1] There are two sources of randomness involved: the location of the break and its size. Section 4.8.3 shows that

$$V_y = \frac{1}{3}\theta^2(1-\theta)^2\delta^2 \qquad (4.12)$$

for a break of magnitude δ at point θ. If a randomly located break is too near a sample endpoint, it is difficult to detect. In terms of (4.12), Ψ, from equation 4.9, is small if θ is near 0 or 1. This causes the rejection probabilities to be smaller for $D_\#$ with a uniformly distributed breakpoint than for $D_{1/2}$. If a random-sized break is small, it is also difficult to detect because Ψ is small if δ^2 is small. The simulations use a normally distributed break size, which puts the highest probability density on the area around a break of size zero, emphasizing this effect. As a consequence, the rejection probabilities are larger for $D_{1/2}$ and $D_\#$ than for $D_{1/2,N}$ and $D_{\#,N}$.

The unit root process, which motivated the original derivation of $\hat{\eta}_\mu$, has rejection probabilities that fall between those for $D_{1/2}$ and $D_\#$ and those for $D_{1/2,N}$ and $D_{\#,N}$. For large γ_T, a break is easier to detect than a unit root even if the break is randomly located as long as the break size is clearly not zero. For the break processes with normally distributed break sizes, the unit root is a little easier to detect. This would likely not be the case if the break size were stochastic, but bounded away from zero.

Overall, however, the differences in rejection probabilities are minimal. While the precise ordering depends on the specifics of the structural break process, the KPSS tests can be considered as effective in detecting structural breaks as they are in detecting in unit roots.

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[1] For large γ, the powers of these tests depend on a complex interaction between the distributions of Ψ and $s_z^2(\ell)$, which are not independent, and the density $g(\hat{\eta}_z)$.

4.6 The KPSS Test for Structural Breaks?

It is not entirely surprising that $\hat{\eta}_\mu$ is sensitive to structural breaks. In fact, $\hat{\eta}_\mu$ is an algebraic special case of the statistic Andrews and Ploberger (1994) propose to test for structural breaks. The general form of the test they propose is a weighted average of LM statistics

$$\int J(\theta) LM(\theta) d\theta. \tag{4.13}$$

where $\theta \in (\theta_1, \theta_2) \subset [0, 1]$ parameterizes the location of the break in the sample and $J(\theta)$ is a weighting function. For a structural break with an unknown breakpoint, Andrews and Ploberger recommend uniform weighting $J(\theta) = 1$, producing the test statistic

$$\int LM(\theta) d\theta \tag{4.14}$$

To compare this statistic with $\hat{\eta}_\mu$, it is necessary to examine the details of the LM statistics. Consider a model

$$y_t = \alpha + \beta d_{i,t} + \varepsilon_t$$

where $d_{i,t}$ is a dummy variable equal to 1 for $t \leq i$ and 0 for $t > i$. The log likelihood function

$$SSR = \frac{1}{2\sigma^2} \sum_{t=1}^{T} (y_t - \alpha - \beta d_{i,t})^2,$$

where σ^2 is the variance of ε_t, has a gradient with respect to β given by

$$\frac{\partial SSR}{\partial \beta} = \frac{1}{\sigma^2} \sum_{t=1}^{T} d_{i,t} e_t = \frac{S_i}{\sigma^2}$$

This is the same S_i that enters the KPSS calculations (4.1). Using $V(S_i) = \theta(1-\theta)\sigma^2$, where $i = [\theta T]$, and substituting the estimate $s^2(\ell)$ for σ^2 yields

$$LM(\theta) = \frac{S_t^2}{T\theta(1-\theta)s^2(\ell)}$$

We can thus write the KPSS statistic as

$$\hat{\eta}_\mu = \int \theta(1-\theta) LM(\theta) d\theta. \tag{4.15}$$

which is (4.13) with the weighting $J(\pi) = \theta(1-\theta)$.

The differences between (4.14) and (4.15) are minimal because $\theta(1-\theta)$ is nearly constant except at the extremes of the $[0, 1]$ range. In fact, Andrews and Ploberger state that the calculations must omit values of θ near 0 or 1 to avoid technical problems. The summation for $\hat{\eta}_\mu$ is not subject to this limitation because the factor $\theta(1-\theta)$ is effectively a continuous counterpart to the sample trimming Andrews and Ploberger recommend.

Indeed, this similarity has been mentioned in passing by Andrews et al. (1996) in the context of the test Nyblom (1989) proposed for parameter changes under martingale alternatives. This general alternative specification encompasses several interesting departures from constancy including a single jump at an unknown point in time (the change-point problem) and slow random variation (a random walk).

4.7 Summary and Conclusions

One way to describe a test is in terms of the assumptions accompanying its derivation. The statistics $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ were originally developed to test for the existence of a unit root component. Lee and Schmidt (1996) refined this description to portray $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ as tests for the existence of a long memory component. Andrews and Ploberger (1994) derive $\hat{\eta}_\mu$ as a test for structural breaks. On this basis, one might describe $\hat{\eta}_\mu$ as a test of the null hypothesis of short memory against an alternative that includes unit roots, fractionally integrated processes, and structural breaks.

One can also characterize a test in terms of what it actually detects. We show that $\hat{\eta}_\mu$ is most sensitive to a cosine function with half a cycle during the sample period and that $\hat{\eta}_\tau$ is most sensitive to a cosine function with a full cycle during the sample period. On this basis, one might describe $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ as tests for cosine functions or, more generally, as tests for realized processes that happen to resemble cosine functions. In the case of $\hat{\eta}_\mu$, the value for a time trend very nearly matches the bound set by the cosine function and the value for a structural break at the sample midpoint is only a little smaller.

We also consider the more practical comparison between the unit root alternative and structural breaks at an unknown breakpoint. While $\hat{\eta}_\mu$ is greater for a pure structural break process with a break in the middle of the sample than for 88% of realized unit roots, an unknown breakpoint uniformly distributed in the sample has an average $\hat{\eta}_\mu$ only somewhat higher than the mean for realized unit roots. The simulation results here show that $\hat{\eta}_\mu$ is equally effective in detecting unit roots and in detecting breaks at an unknown breakpoint.

An important caution is implied by the fact that $\hat{\eta}_\mu$ is described as a test for unit roots by KPSS and as a test for structural breaks by Andrews and Ploberger. If $\hat{\eta}_\mu$ is large enough to reject the null hypothesis of short memory, it would be entirely misleading to cite KPSS and claim discovery of a unit root just as it would be misleading to cite Andrews and Ploberger and claim evidence of a structural break. The only appropriate conclusion is a rejection of the null hypothesis of short memory.

4.8 Proofs and Calculations

4.8.1 Proof of Proposition 2

We seek to maximize

$$\hat{\eta}_\mu = \frac{\int_0^1 S_t^2 dt}{s^2} \tag{4.16}$$

where

$$S_t = \int_0^t y(s)ds - t\int_0^1 y(r)dr \tag{4.17}$$

and

$$s^2 = \int_0^1 \left(y(t) - \int_0^1 y(r)dr\right)^2 dt.$$

Our strategy is as follows. Differentiating (4.16) with respect to $y(a)$ yields a condition that $y(a)$ must satisfy. Differentiating this condition with respect to a then yields a differential equation that $y(a)$ must satisfy for all a. Differentiating a second time yields a second-order differential equation that does not involve the integrals in (4.17).

We differentiate (4.16) with respect to $y(a)$ using the Frechet derivatives

$$\frac{\partial S_t}{\partial y(a)} = \left\{ \begin{array}{cc} 1-t, & t \geq a \\ -t, & t < a \end{array} \right\}$$

and

$$\frac{\partial s^2}{\partial y(a)} = 2\left(y(a) - \int_0^1 y(r)dr\right)$$

to obtain the derivative

$$\frac{1}{2}\frac{\partial \hat{\eta}_\mu}{\partial y(a)} = \frac{\int_0^a (-t)S_t dt}{s^2} + \frac{\int_a^1 (1-t)S_t dt}{s^2} - \frac{\int_0^1 S_t^2 dt}{(s^2)^2}\left(y(a) - \int_0^1 y(r)dr\right).$$

Equating this to zero yields

$$\frac{(y(a) - \int_0^1 y(r)dr)}{s^2} = \frac{\int_0^a (-t) S_t dt}{\int_0^1 S_t^2 dt} + \frac{\int_a^1 (1-t) S_t dt}{\int_0^1 S_t^2 dt}$$

Differentiating both sides with respect to a yields

$$\frac{\partial y(a)/\partial a}{s^2} = -\frac{a S_a}{\int_0^1 S_t^2 dt} - \frac{(1-a) S_a}{\int_0^1 S_t^2 dt}.$$

which can be simplified to

$$\frac{S_a}{\int_0^1 S_t^2 dt} = -\frac{\partial y(a)/\partial a}{s^2}.$$

We thus have that the maximum KPSS statistic is attained for data following the recursion

$$\frac{\partial y(a)}{\partial a} = -\widehat{\eta}_\mu^{-1} S_a$$

Substitute the definition for S_a

$$\frac{\partial y(a)}{\partial a} = -\widehat{\eta}_\mu^{-1} \left(\int_0^a y(t)dt - a \int_0^1 y(r)dr \right).$$

We then need to find solutions to

$$\frac{\partial^2 y(a)}{\partial a^2} = -\widehat{\eta}_\mu^{-1} \left(y(a) - \int_0^1 y(r)dr \right) \tag{4.18}$$

The residuals

$$e(a) = y(a) - \int_0^1 y(r)dr$$

have the property that

$$\frac{\partial^2 e(a)}{\partial a^2} = \frac{\partial^2 y(a)}{\partial a^2}$$

We can thus rewrite (4.18) as

$$\frac{\partial^2 e(a)}{\partial a^2} = -\widehat{\eta}_\mu^{-1} e(a). \tag{4.19}$$

The solutions to this differential equation are of the form

$$y(t) = y(0)\cos(kt + c).$$

where

$$k^2 = \widehat{\eta}_\mu^{-1}. \tag{4.20}$$

We are, therefore, looking for solutions to (4.19) with

$$\hat{\eta}_\mu = \frac{\int_0^1 \left(\int_0^a cos(kt+c)dt\right)^2 da}{\int_0^1 cos^2(kt+c)dt - \left[\int_0^1 cos(kt+c)dt\right]^2}. \tag{4.21}$$

Using

$$\int_0^a cos(kt+c)dt = k^{-1}sin(ka+c) - k^{-1}sin(c),$$

the numerator is

$$\int_0^1 \left(\int_0^1 cos(kt+c)dt\right)^2 = \int_0^1 k^{-2}sin^2(kt+c)dt - 2\int_0^1 k^{-2}sin(kt+c)sin(c)dt + \int_0^1 k^{-2}sin^2(c)dt$$

$$= \frac{k^{-2}}{2} - \frac{k^{-3}}{2}sin(k)cos(k) + 2k^{-2}sin(c)(cos(k+c) - cos(c)) + k^{-2}sin^2(c).$$

The denominator is the sum of

$$\int_0^1 cos^2(kt+c)dt = \tfrac{1}{2} + \tfrac{1}{2}k^{-1}sin(k+c)cos(k+c) - \tfrac{1}{2}k^{-1}sin(c)cos(c)$$

and

$$\left[\int_0^1 cos(kt+c)dt\right]^2 = k^{-2}sin^2(k+c)$$

Putting all this together, we have

$$\hat{\eta} = k^{-2}\frac{\frac{1}{2} - \frac{1}{2}k^{-1}sin(k+c)cos(k+c) + 2sin(c)(cos(k+c) - cos(c)) + sin^2(c)}{\frac{1}{2} + \frac{1}{2}k^{-1}sin(k+c)cos(k+c) - \frac{1}{2}k^{-1}sin(c)cos(c) - k^{-2}sin^2(k+c)}$$

We need to find the values of k and c for which the fraction on the right equals 1. This happens if $sin(c) = 0$ and $sin(k+c) = 0$ implying that the maximum occurs for $c = 0$ and k in the set of values $\pi, 2\pi, 3\pi, \ldots$ Checking the values of (4.21) shows that the $k = \pi$ determines the largest possible KPSS statistic $\pi^{-2}T$.

The bound for $\hat{\eta}_\tau$ follows directly from this result. The statistic $\hat{\eta}_\tau$ can be calculated for a variable $w(t)$ by detrending that variable to produce residuals denoted $y(t)$ and then calculating $\hat{\eta}_\mu$ for $y(t)$. The bound on this $\hat{\eta}_\mu$ is determined by the above analysis with the further condition that $y(t)$ must be detrended. The choice $k = 2\pi$ produces the largest $\hat{\eta}_\mu$ with this property because $y(t)$ is trend-free for $k = 2\pi, 4\pi, \ldots$. □

4.8.2 Proof of Proposition 3

We can write the KPSS statistic as

$$\hat{\eta}_z = \frac{T^{-2}\sum_{t=1}^{T}(S_{x\,t} + \gamma_T S_{y\,t})^2}{s_z^2(\ell)}$$

or as

$$\hat{\eta}_z = \frac{T^{-2}\sum_{t=1}^{T} S_{x\,t}^2 + 2\gamma_T T^{-2}\sum_{t=1}^{T} S_{x\,t}S_{y\,t} + \gamma_T^2 T^{-2}\sum_{t=1}^{T} S_{y\,t}^2}{s_x^2(\ell) + s_z^2(\ell) - s_x^2(\ell)}$$

It will prove convenient to reorganize these terms as

$$\hat{\eta}_z = \frac{\hat{\eta}_x + A}{1 + B}$$

where

$$A = 2\gamma_T \frac{T^{-2}\sum_{t=1}^{T} S_{x\,t}S_{y\,t}}{s_x^2(\ell)} + \gamma_T^2 \frac{T^{-2}\sum_{t=1}^{T} S_{y\,t}^2}{s_x^2(\ell)}$$

and

$$B = \frac{s_z^2(\ell) - s_x^2(\ell)}{s_x^2(\ell)}$$

This allows to express the rejection probability as

$$Pr(\hat{\eta}_z > c^*) = Pr(\hat{\eta}_x > c^*(1 + B) - A)$$

Integrating over the densities for x and y yields.

$$Pr(\hat{\eta}_z > c^*) = \iint G(c^*(1 + B) - A) f(y) f(x) \partial y \partial x$$

where $G(c) = Pr(\hat{\eta}_x > c)$. Using $Pr(\hat{\eta}_x > c^*) = \alpha^*$ and $\partial G(c)/\partial c = -g(c)$, we can expand $G(c^*(1 + B) - A) - G(c^*)$ to obtain

$$Pr(\hat{\eta}_z > c^*) - \alpha^* \cong g(c^*) \iint (A - c^* B) f(y) f(x) \partial y \partial x \tag{4.22}$$

We can analyze A and B as follows. The fact that x_t and y_t are independent implies that

$$\int\int \gamma_T^{-2} T^{-3} \sum_{t=1}^{T} S_{x\,t}S_{y\,t} f(y) f(x) dy dx = 0.$$

The definition of V_y then yields

$$\int T^{-3}\sum_{t=1}^{T} S_{y\,t}^2 f(y) dy = V_y.$$

Together, these give us

$$\frac{\iint A f(y) f(x) \partial y \partial x}{\gamma_T^2 T} = \frac{V_y}{\gamma_T^2 T \sigma_x^2}.$$

The fact that x_t and y_t are independent also implies that

$$s_z^2(f) = s_x^2(f) + \gamma_T^2 s_y^2(f).$$

From this, we obtain

$$\frac{\iint B f(y) f(x) \partial y \partial x}{\gamma_T^2 T} = \frac{s_y^2(f)}{T s_x^2(f)} \to 0.$$

We can now substitute these results into (4.22) to obtain

$$Pr(\hat{\eta}_z > c^*) - \alpha^* \to g(c^*) \frac{V_y}{\gamma_T^2 T \sigma_x^2}.$$

□

4.8.3 Notes on Calculations for Tables 4.1 and 4.2

Trends

TT: $y_t = t$

$$s^2 = \int_0^1 (t - \tfrac{1}{2})^2 dt = \tfrac{1}{12}$$

$$S_t = \int_0^t (t - \tfrac{1}{2}) dt = \tfrac{1}{2}(t - \tfrac{1}{2})^2 - \tfrac{1}{8}$$

$$\begin{aligned} \int_0^1 S_t^2 dt &= \int_0^1 \left[\tfrac{1}{4}(t - \tfrac{1}{2})^4 - \tfrac{1}{8}(t - \tfrac{1}{2})^2 + \tfrac{1}{64}\right] dt \\ &= \tfrac{1}{20}(t - \tfrac{1}{2})^5 - \tfrac{1}{24}(t - \tfrac{1}{2})^3 + \tfrac{1}{16} t \,\Big|_0^1 \\ &= \tfrac{1}{320} - \tfrac{1}{96} + \tfrac{1}{64} = \tfrac{3-10+15}{960} = \tfrac{1}{120} \end{aligned}$$

TT2: $y_t = t$ for $0 \le t \le 1/2$; $y_t = 1/2 - t$ for $1/2 \le t \le 1$.

$$s^2 = 2\int_0^{1/2} (t - \tfrac{1}{4})^2 dt = \tfrac{1}{48}$$

From 0 to 1/2,

$$S_t = \tfrac{1}{2}(t - \tfrac{1}{4})^2 - \tfrac{1}{32}$$

$$\int_0^1 S_t^2 dt = 2\int_0^{1/2} \left(\tfrac{1}{4}(t - \tfrac{1}{4})^4 - \tfrac{1}{32}(t - \tfrac{1}{4})^2 + \tfrac{1}{1024}\right) dt = \tfrac{1}{1920}$$

Breaks

We derive the result for an arbitrary break located at θ. $y_t = \delta$ for $t \leq [\theta T]$; $y_t = 0$ for $t > [\theta T]$.

$$s^2 = \theta[(1-\theta)\delta]^2 + (1-\theta)[\theta\delta]^2$$

$$s^2 = \theta(1-\theta)\delta^2$$

$$\int_0^1 S_t^2 dt = \int_0^\theta [(1-\theta)\delta t]^2 dt + \int_\theta^1 [\theta\delta(1-t)]^2 dt = \tfrac{1}{3}(1-\theta)^2\theta^2\delta^2$$

D$_{1/2}$: Do the obvious substitution.

D$_\theta$: Break with θ uniformly distributed on [0,1].

$$\int_0^1 \theta(1-\theta)d\theta = \tfrac{1}{6}$$

$$\int_0^1 \theta^2(1-\theta)^2 d\theta = \tfrac{1}{30}$$

D$_{1/2,N}$ and D$_{\theta,N}$: Use the expectation of the square.

D2: $y_t = 1$ for (0, 1/4) and (3/4, 1); $y_t = -1$ for (1/2, 3/4).

$$s^2 = \tfrac{1}{4}$$

$$\int_0^1 S_t^2 dt = 4\int_0^{1/4} (\tfrac{1}{2}t)^2 dt = \tfrac{1}{192}$$

Unit Roots

UR: $y_t = y_{t-1} + \varepsilon_t$, ε_t i.i.d., $E(\varepsilon_t) = 0$, $V(\varepsilon_t) = 1$. We will use the two lemmas:

$$E\left(\int_0^a y_t^2 dt\right) = \int_0^a t\,dt \tag{4.23}$$

$$E\left(\left(\int_0^a y_t dt\right)^2\right) = \int_0^a t^2 dt \tag{4.24}$$

$$E(s^2) = \int_0^1 y_t^2 dt - \left(\int_0^1 y_t dt\right)^2$$

$$= \int_0^1 t\,dt - \int_0^1 t^2 dt = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}$$

$$E\left[\left(\int_0^t y_i di\right)\left(\int_t^1 y_i di\right)\right] = (1-t)\int_0^t i di \tag{4.25}$$

$$E\left(\int_t^1 y_i di\right)^2 = \int_0^{1-t} i^2 di + t(1-t)^2 \tag{4.26}$$

$$S_t = \int_0^t y_i di - t\int_0^1 y_i di$$

Rearrange as

$$S_t = (1-t)\int_0^t y_i di - t\int_t^1 y_i di$$

$$S_t^2 = (1-t)^2\left(\int_0^t y_i di\right)^2 - 2t(1-t)\left(\int_0^t y_i di\right)\left(\int_t^1 y_i di\right) + t^2\left(\int_t^1 y_i di\right)^2$$

Use independence (4.25) of the middle term.

$$E(S_t^2) = (1-t)^2\int_0^t i^2 di - 2t(1-t)^2\int_0^t i di + t^2\int_0^{1-t} i^2 di + t^3(1-t)^2$$

$$E(S_t^2) = (1-t)^2t^3/3 - 2t(1-t)^2t^2/2 + t^2(1-t)^3/3 + t^3(1-t)^2$$

$$E(S_t^2) = (1-t)^2t^2[t/3 - t + (1-t)/3 + t]$$

$$E(S_t^2) = (1-t)^2t^2[1/3]$$

$$E\left(\int_0^1 S_t^2\right) = \frac{1}{30}\int_0^1 (1-t)^2 t^2 dt = \frac{1}{90}$$

$$\int_0^1 (1-t)^2t^2 dt = \frac{1}{3} - \frac{2}{4} + \frac{1}{5} = \frac{1}{30}$$

UR2: These results are derived from simulations.

Table 4.1: Expected Values for $\hat{\eta}_\mu$ and $s_y^2(0)$ for V_y

Process	Description	$\hat{\eta}_\mu$	$s_y^2(0)$
COS	$y_t = 2^{1/2}\pi cos(\pi t/T)$	$\pi^{-2}T$	$\pi^2 = 9.87\ldots$
TT	$y_t = \beta t,\ \beta = 120^{1/2}T^{-1}$	$\frac{1}{10}T$	10
$D_{1/2}$	$y_t = \delta d_{1/2,t},\ \delta = 48^{1/2}$	$\frac{1}{12}T$	12

Process	Description	$E(\hat{\eta}_\mu)$	$E(s_y^2(0))$
$D_{1/2,N}$	$y_t = \delta d_{1/2,t},\ \delta \sim N(0,48)$	$\frac{1}{12}T$	12
D_θ	$y_t = \delta d_{\theta,t},\ \delta = 90^{1/2}$	$\frac{1}{18}T$	16
$D_{\theta,N}$	$y_t = \delta d_{\theta,t},\ \delta \sim N(0,90)$	$\frac{1}{18}T$	16
UR	$y_t = y_{t-1} + \varepsilon_t,\ \sigma_\varepsilon^2 = 90/T$	$\frac{1}{20}T$	15

$d_{\theta,t} = \{1 \text{ for } t < [\theta T],\ 0 \text{ otherwise}\}$. $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$.

Table 4.2: Expected Values for $\hat{\eta}_\tau$ and $s_y^2(0)$ for V_y

Process	Description	$\hat{\eta}_\tau$	$s_y^2(0)$
COS2	$y_t = 2^{3/2}\pi cos(2\pi t/T)$	$(2\pi)^{-2}T$	$(2\pi)^2$
TT2	$y_t = \begin{cases} (480)^{1/2}t & \text{for } t \leq \frac{1}{2} \\ (480)^{1/2}(\frac{1}{2} - t) & \text{for } t > \frac{1}{2} \end{cases}$	$\frac{1}{40}T$	40
D2	$y_t = \begin{cases} -48^{1/2} & \text{for } t \in [\frac{1}{4}T, \frac{3}{4}T] \\ +48^{1/2} & \text{otherwise} \end{cases}$	$\frac{1}{48}T$	48

Process	Description	$E(\hat{\eta}_\tau)$	$E(s_y^2(0))$
UR2	$y_t = y_{t-1} + \varepsilon_t, \sigma^2 = 1200/T$	$0.0109T$	80

Table 4.3: Distributions of $\hat{\eta}_\mu$ and $\hat{\eta}_\tau$ for a Pure Unit Root

α	$Pr(\hat{\eta}_\mu < \alpha\pi^{-2}T)$	$Pr(\hat{\eta}_\tau < \alpha(2\pi)^{-2}T)$
0.10	0.051	0.018
0.20	0.180	0.145
0.30	0.301	0.320
0.40	0.402	0.494
0.50	0.490	0.645
0.60	0.592	0.764
0.70	0.697	0.869
0.80	0.820	0.951
0.90	0.946	0.996

Table 4.4: Rejection Probabilities for $\hat{\eta}_\mu$ where $y_t = x_t + \gamma z_t$

T = 50

gamma	COS	TT	$D(\frac{1}{2})$	$D(\frac{1}{2}, N)$	$D(\theta)$	$D(\theta, N)$	UR
0.00	0.050	0.051	0.054	0.051	0.048	0.052	0.057
0.02	0.070	0.073	0.074	0.073	0.075	0.072	0.072
0.04	0.143	0.138	0.138	0.137	0.145	0.143	0.142
0.06	0.253	0.249	0.250	0.232	0.251	0.218	0.224
0.08	0.412	0.414	0.408	0.328	0.384	0.288	0.325
0.10	0.583	0.583	0.566	0.413	0.529	0.369	0.399
0.12	0.739	0.737	0.733	0.489	0.633	0.424	0.481
0.14	0.860	0.856	0.851	0.541	0.708	0.469	0.536
0.16	0.935	0.939	0.932	0.588	0.766	0.511	0.595
0.18	0.974	0.975	0.972	0.620	0.796	0.545	0.640
0.20	0.991	0.990	0.989	0.659	0.819	0.572	0.679

T = 200

gamma	COS	TT	$D(\frac{1}{2})$	$D(\frac{1}{2}, N)$	$D(\theta)$	$D(\theta, N)$	UR
0.00	0.052	0.049	0.051	0.049	0.051	0.054	0.051
0.01	0.073	0.074	0.074	0.073	0.073	0.072	0.070
0.02	0.140	0.145	0.144	0.145	0.143	0.143	0.138
0.03	0.263	0.261	0.264	0.233	0.259	0.222	0.239
0.04	0.411	0.422	0.413	0.329	0.405	0.297	0.338
0.05	0.591	0.595	0.592	0.412	0.536	0.367	0.431
0.06	0.747	0.742	0.749	0.482	0.651	0.423	0.500
0.07	0.873	0.878	0.869	0.541	0.718	0.475	0.574
0.08	0.939	0.939	0.939	0.590	0.768	0.517	0.630
0.09	0.978	0.977	0.977	0.624	0.796	0.549	0.681
0.10	0.992	0.993	0.993	0.658	0.822	0.580	0.720

T = 800

gamma	COS	TT	$D(\frac{1}{2})$	$D(\frac{1}{2}, N)$	$D(\theta)$	$D(\theta, N)$	UR
0.000	0.052	0.050	0.048	0.053	0.053	0.050	0.050
0.005	0.073	0.073	0.074	0.072	0.070	0.071	0.070
0.010	0.143	0.141	0.140	0.144	0.144	0.138	0.139
0.015	0.270	0.268	0.267	0.244	0.256	0.221	0.234
0.020	0.429	0.424	0.418	0.329	0.399	0.300	0.339
0.025	0.596	0.606	0.597	0.420	0.539	0.374	0.435
0.030	0.751	0.758	0.756	0.485	0.639	0.426	0.514
0.035	0.872	0.873	0.875	0.559	0.714	0.482	0.586
0.040	0.939	0.944	0.942	0.609	0.767	0.513	0.634
0.045	0.981	0.975	0.981	0.631	0.802	0.554	0.688
0.050	0.993	0.994	0.994	0.661	0.813	0.577	0.732

Table 4.5: Rejection Probabilities for $\hat{\eta}$- where $y_t = x_t + \gamma z_t$

T = 50

gamma	COS2	TT2	D2	UR2
0.00	0.052	0.057	0.053	0.055
0.01	0.076	0.077	0.075	0.076
0.02	0.138	0.142	0.137	0.129
0.03	0.246	0.254	0.240	0.221
0.04	0.396	0.396	0.390	0.332
0.05	0.564	0.557	0.555	0.433
0.06	0.714	0.716	0.714	0.512
0.07	0.843	0.847	0.836	0.584
0.08	0.924	0.926	0.916	0.647
0.09	0.969	0.968	0.966	0.703
0.10	0.990	0.990	0.988	0.746

T = 200

gamma	COS2	TT2	D2	UR2
0.00	0.055	0.056	0.055	0.049
0.01	0.072	0.068	0.078	0.078
0.01	0.139	0.136	0.134	0.139
0.02	0.247	0.251	0.252	0.233
0.02	0.402	0.398	0.402	0.343
0.03	0.568	0.574	0.575	0.468
0.03	0.730	0.727	0.723	0.563
0.04	0.854	0.849	0.850	0.650
0.04	0.935	0.935	0.932	0.722
0.05	0.974	0.974	0.971	0.771
0.05	0.991	0.991	0.991	0.829

T = 800

gamma	COS2	TT2	D2	UR2
0.000	0.056	0.050	0.054	0.052
0.003	0.069	0.081	0.074	0.066
0.005	0.137	0.138	0.134	0.141
0.007	0.250	0.251	0.253	0.246
0.010	0.407	0.411	0.402	0.361
0.013	0.578	0.572	0.580	0.477
0.015	0.729	0.732	0.732	0.583
0.018	0.858	0.856	0.857	0.659
0.020	0.934	0.934	0.936	0.737
0.022	0.976	0.972	0.973	0.790
0.025	0.992	0.990	0.992	0.841

Figure 4.1: Power Functions for $\hat{\eta}_\mu$ where $y_t = x_t + \gamma z_t$

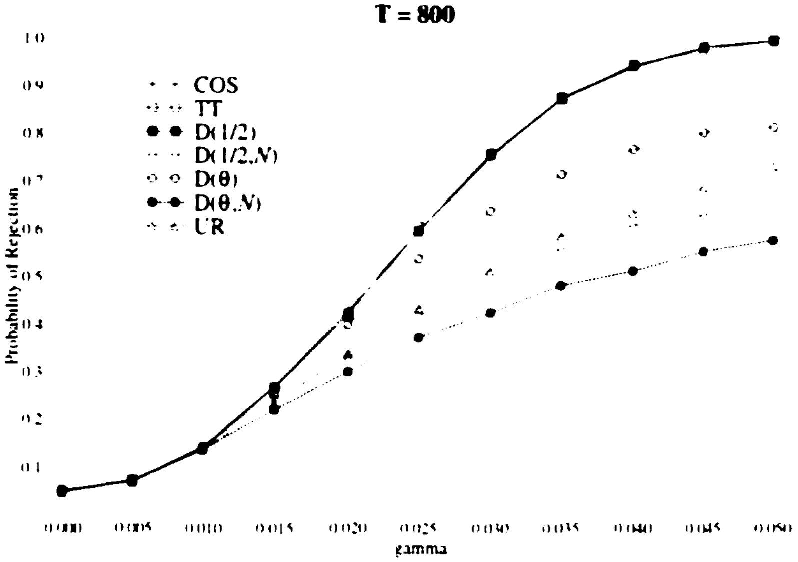
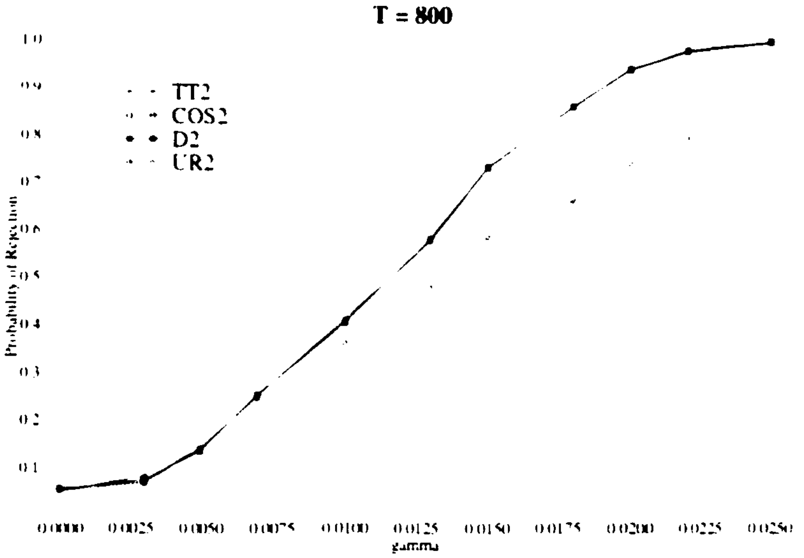


Figure 4.2: Power Functions for $\hat{\eta}_\tau$ where $y_t = x_t + \gamma z_t$



Chapter 5

Distinguishing Between Structural Change and Unit Roots

We propose a two-stage procedure for distinguishing between structural change and unit roots. Given a first-stage test result that rejects the stationarity hypothesis, we conduct a second-stage test on sub-samples to consider whether the entire sample appears to be nonstationary or whether the nonstationarity can be attributed to limited sub-samples. This second-stage test is shown to be effective in discriminating between structural change and unit roots.

5.1 Introduction

This chapter expands upon the results of Chapter 3, which provided a unifying algebraic framework for tests for structural change and tests for nonstationarity, illustrating the fundamental similarities linking the two disparate strands of the literature. The practical implication of these similarities is that the differences among the tests are minimal for a range of alternatives, including structural change, unit roots, and fractional integration.

The results also identify an important pitfall. It is quite misleading to conclude that a process is a unit root if you reject using the KPSS test, but that it is a structural change if you reject using the algebraically equivalent Andrews and Ploberger formulation. The best you can say with either test is that, if you reject the null of short memory, then the process is not short memory.

Structural change and other forms of nonstationarity are not, however, identical, and we introduce a new test that effectively discriminates between the two forms of long memory. We can express the general approach we take in terms of a two-stage hypothesis testing structure. The question of primary interest in Stage I centers on the null and alternative hypotheses

$$H_0 : \qquad y_t = \varepsilon_t$$

$$H_a : \qquad y_t = \varepsilon_t + w_t$$

where ε_t is a short memory process and w_t is a long memory process. We partition the alternative H_a into two Stage II hypotheses

$$H_n : \qquad y_t = \varepsilon_t + d_t$$

$$H_\infty : \qquad y_t = \varepsilon_t + r_t ,$$

where r_t is a unit root process and d_t is a long memory process that is not a unit root. In particular, we are interested in a prominent member of H_n, a sub-set we call H_1. For H_1, d_t is a structural break process, with a single permanent change somewhere in the sample. We choose our notation for the Stage II hypotheses in this way to highlight the recent work relating structural change and long memory.[1] In this sense, a unit root can be thought of as a structural break process with a break in every period. Hence our use of the notation H_∞ for the unit root alternative.

We propose a procedure for distinguishing between H_n and H_∞. Given a rejection of short memory using a statistic Λ, we propose calculating Λ for sub-samples of the data. The procedure is motivated by the Stage II alternative H_1. The basic idea is that the subsample results should show a localized rejection in the case of a single structural break and widespread rejection in the case of a unit root.

We show that the test is consistent and that the empirical performance of this two-stage test is surprisingly good in distinguishing between H_1 and H_∞. If the KPSS (or Andrews) test rejects the null of stationarity, the probability that the second stage test correctly classifies the type of nonstationarity is reasonably high. On the other hand, the Stage II test is completely meaningless without a first-stage rejection. We also look at power for two additional members

[1] See Parke (1999), Diebold and Inoue (2001), and Granger and Hyung (1999).

of H_n, an interior break and a fractionally integrated process. While our test performs well for the interior break, it seems to have very little power against an $I(d)$ process, particularly when d is in the non-stationary range. This agrees with the results of Lee and Amsler (1997) who find that the KPSS test has the same order in probability under both a unit root and non-stationary fractional integration. Further, our test looks for heterogeneity while $I(d)$ processes are homogeneous.

We will organize our discussion as follows. Section 5.2 introduces the test between unit roots and structural change. Section 5.3 derives the asymptotic distribution under H_∞ and Section 5.4 provides critical values. Section 5.5 derives the normalized break response functions for several statistics of interest and Section 5.6 discusses consistency. Section 5.7 gives evidence on the power of the tests, and we summarize our results in Section 5.8.

5.2 A Subsample Test

If the Stage I test rejects the short memory hypothesis, then the next natural question is whether the evidence can plausibly be accounted for by a unit root explanation (H_∞) or whether it points to another member of H_A. Here, we propose a procedure for distinguishing between these two cases that is motivated by a specific Stage II alternative, the hypothesis H_1 that there is a single break in the sample.

Consider the unit root model

$$z_t = z_{t-1} + u_t \tag{5.1}$$

where u_t is a stationary mean zero process with long-run variance σ_u^2 We assume u_t satisfies the regularity conditions of Assumption 2.1 in Phillips (1987), which requires the existence of absolute moments of order β, for some $\beta > 2$, and α-mixing with mixing coefficients α_m such that $\sum_{m=1}^{\infty} \alpha_m^{1-2/\beta} < \infty$. Let $M = \{1, \ldots, T-k\}$, $1 < k < T$, denote the set of subsample starting points. For $m \in M$ define the sub-sample residuals from a regression of $z_t \in [m, m+k]$ on a constant as

$$e_{m\,m+n} = z_{m+n} - k^{-1} \sum_{i=m}^{m+k} z_i, \qquad 1 \leq n \leq k. \tag{5.2}$$

The sub-sample residual partial sum $S_{m\,m+[rk]}$, which will form the basis for our statistics, is given by

$$S_{m,m+n} = \sum_{i=1}^{n} \epsilon_{m,m+i}, \qquad 1 \leq n \leq k. \tag{5.3}$$

Let $\lambda(m,k)$ denote a statistic with a null hypothesis of short memory applied to a subsample starting in period m and of length k. (The tests discussed in the previous chapters would thus be denoted $\lambda(1,T)$.) For example, $\lambda(m,k)$ could be a KPSS statistic (or an Andrews statistic) calculated over a subsample of the data. Our proposed subsample statistic for testing the hypothesis $H_{\sim}$ of a unit root process is given by

$$\Lambda_i(k/T) = \frac{\operatorname{avg}_{m \in M} \lambda_i(m,k)}{\sup_{m \in M} \lambda_i(m,k)} \tag{5.4}$$

where the i denotes the particular underlying stability statistic used in the computation of $\Lambda(k/T)$. Several candidates for $\Lambda(k/T)$ present themselves from the discussion in Chapter 3.

$$(\text{AHBT}(0.5)) \qquad \lambda_1(m,k) = 4k^{-2} S^2_{m,m+k/2}$$

$$(\text{AvgS}) \qquad \lambda_2(m,k) = k^{-3} \sum_{n=1}^{k} S^2_{m,m+n}$$

$$(\text{SupS}) \qquad \lambda_3(m,k) = k^{-2} \sup_{n \in [1,k]} S^2_{m,m+n}$$

$$(\text{AvgLM}(\pi_0)) \qquad \lambda_4(m,k,\pi_0) = [k^3(1-2\pi_0)]^{-1} \sum_{n=\pi_0 k}^{(1-\pi_0)k} \frac{S^2_{m,m+n}}{\frac{n}{k}\left(1-\frac{n}{k}\right)}$$

$$(\text{SupS}(\pi_0)) \qquad \lambda_5(m,k) = k^{-2} \sup_{n \in [\pi_0 k,(1-\pi_0)k]} \frac{S^2_{m,m+n}}{\frac{n}{k}\left(1-\frac{n}{k}\right)}$$

In a Stage I test, the statistics above are normalized by dividing by an estimator of the long-run variance of u_t in equation (5.1). Although under $H_{\sim}$ the long-run variance does not exist, the estimator for any finite T impacts the numerator and denominator of the asymptotic distribution of each statistic in the same way and thus its effect cancels out. The distributions of $\Lambda_i(q)$, $i = 1,\ldots,5$ under $H_{\sim}$ are given in the following section.

Motivated by H_1, the intuitive basis for this test is as follows. If the true model has a single structural break somewhere in the sample, then the large values of $\lambda(m,k)$ will tend to be for the sub-samples that overlap the structural break. The average $\lambda(m,k)$, on the other hand, will reflect the extent to which all the sub-samples support the hypothesis of nonstationarity. Since most of the sub-samples will not include a structural break, the average will be much less supporting of the nonstationarity conclusion than is the supremum. For a unit root, where the sub-samples are in fact generated by a homogeneous process, the ratio of the average to the supremum will tend to be a larger number.

5.3 Asymptotic Distribution Under H_∞

This section establishes the convergence of our Stage II statistics to their asymptotic distributions. We use these results to justify the simulation of critical values in the following section.

We will first define some notation for sub-sample Wiener processes. Then we will show that the partial sums from sub-samples converge to these Wiener processes. Finally, we will put these together to find the distributions of our statistics.

We begin by gathering some useful definitions that hold for a contiguous sub-sample of length $q \in [0,1]$ beginning at a point b, $b \in B$ where $B = [0, 1-q]$. Define the sub-sample Wiener process $W^*(b,q,r)$ for $r \in [0,1]$ as:

$$W^*(b,q,r) = q^{-1/2}\left[W(b+rq) - W(b)\right]$$

where $W(\cdot)$ is a standard Wiener process. Then the demeaned sub-sample Wiener process $\underline{W}^*(b,q,r)$ is given by

$$\underline{W}^*(b,q,r) = W^*(b,q,r) - \int_0^1 W^*(b,q,s)ds.$$

Finally, denote the continuous time analog of the sub-sample partial sum of a demeaned Wiener process by

$$Q(b,q,r) = \int_b^{b+rq} \underline{W}^*(b,q,a)da.$$

Since u_t in (5.1) is stationary, the following invariance principle for the convergence of $z_{m,m+[rk]}$ to a sub-sample Wiener process holds as $T \to \infty$. For k/T constant and $r \in [0,1]$,

$$k^{-1/2} z_{m,m+[rk]} = k^{-1/2} \sum_{j=m}^{m+[rk]} u_j \to \sigma_u^* W^*(b,q,r). \qquad (5.5)$$

Here $\to$ denotes convergence in distribution and $[\cdot]$ denotes the integer part.

The following lemma provides the building blocks for the asymptotic distribution of $\Lambda_i(q)$.

Lemma 2. *Given the regularity conditions on u_t and the sub-sample invariance principle above*

(i) $k^{-3/2} z_{m,m+[rk]} \to \sigma_u^* \int_0^r W^*(b,q,s)ds$

(ii) $k^{-1/2} \epsilon_{m,m+[rk]} \to \sigma_u^* \underline{W}^*(b,q,r)$

(iii) $k^{-3/2} S_{m,m+[rk]} \to \sigma_u^* Q(b,q,r)$

(iv) $k^{-3} S^2_{m,m+[rk]} \to \sigma_u^{*2} Q(b,q,r)^2$

Proof. From (5.5) above, (i) follows directly. To establish (ii), note

$$k^{-1/2}\epsilon_{m+[rk]} = k^{-1/2} z_{m,m+[rk]} - k^{-1}\sum_{i=1}^{k} k^{-1/2} z_{m,m+i}$$

$$\to \sigma_u^* W^*(b,q,r) - \sigma_u^* \int_0^1 W^*(b,q,r)\,dr = \sigma_u^* \underline{W}^*(b,q,r),$$

which proves (ii). (iii) follows from (ii) as

$$k^{-3/2} S_{m,m+[rk]} = k^{-1}\sum_{i=m}^{m+[rk]} k^{-1/2}\epsilon_{m,m+[rk]} \to \sigma_u^* \int_b^{b+rq} \underline{W}^*(b,q,r)da.$$

(iv) then follows from (iii) because

$$k^{-3} S^2_{m,m+[rk]} = (k^{-3/2} S_{m,m+[rk]})^2 \to \sigma_u^{*2} Q(b,q,r)^2,$$

which yields the result. □

Lemma 2 allows us to find the distribution of Λ under a unit root. The distribution for several different sub-sample statistics of interest follow.

Proposition 4 (Λ-AHBT(1/2)).

$$\Lambda_1(q) \longrightarrow \frac{\int_0^{1-q} Q(b,q,\frac{1}{2})^2 db}{\sup_{b\in B} Q(b,q,\frac{1}{2})^2} \tag{5.6}$$

Proof. Notice

(i) $(T-k)^{-1}k^{-3}\sum_{m=1}^{T-k} 4S^2_{m,m+\frac{k}{2}} \to 4\sigma_u^{*2}\int_0^{1-q} Q(b,q,\frac{1}{2})^2\,db$

(ii) $k^{-3}\sup_{m\in M}(4S^2_{m,m+\frac{k}{2}}) \to 4\sigma_u^{*2}\sup_{b\in B} Q(b,q,\frac{1}{2})^2$

follow from the application of Lemma 2. The result follows directly. □

Proposition 5 (Λ-AvgS).

$$\Lambda_2(q) \longrightarrow \frac{\int_0^{1-q}\int_0^1 Q(b,q,r)^2\,dr\,db}{\sup_{b\in B}\int_0^1 Q(b,q,r)^2\,dr} \tag{5.7}$$

Proof. Notice

(i) $k^{-4}\sum_{n=1}^{k} S^2_{m,m+[rk]} \to \sigma_u^{*2}\int_0^1 Q(b,q,r)^2\,dr$

(ii) $(T-k)^{-1}k^{-4}\sum_{m=1}^{T-k}\sum_{n=1}^{k} S^2_{m,m+[rk]} \to \sigma_u^{*2}\int_0^{1-q}\int_0^1 Q(b,q,r)^2\,dr\,db$

(iii) $k^{-4}\sup_{m\in M}\sum_{n=1}^{k} S^2_{m,m+[rk]} \to \sigma_u^{*2}\sup_{b\in B}\int_0^1 Q(b,q,r)^2\,dr$

follow from the application of Lemma 2. The result follows directly. □

Proposition 6 (Λ-SupS).

$$\Lambda_3(q) \longrightarrow \frac{\int_0^{1-q}\sup_{s\in[0,1]} Q(b,q,r)^2\,db}{\sup_{b\in B}\sup_{s\in[0,1]} Q(b,q,r)^2} \tag{5.8}$$

Proof. Notice

(i) $(T-k)^{-1}k^{-3}\sum_{m=1}^{T-k}\sup_{n\in[1,k]} S^2_{m,m+[rk]} \to \sigma_u^{*2}\int_0^{1-q}\sup_{r\in[0,1]} Q(b,q,r)^2\,db$

(ii) $k^{-3}\sup_{m\in M}\sup_{n\in[1,k]} S^2_{m,m+[rk]} \to \sigma_u^{*2}\sup_{b\in B}\sup_{r\in[0,1]} Q(b,q,r)^2$

follow from the application of Lemma 2. The result follows directly. □

Proposition 7 (Λ-AvgLM(π_0)).

$$\Lambda_4(q) \longrightarrow \frac{\int_0^{1-q}\int_{\pi_0}^{1-\pi_0} Q(b,q,r)^2\,dr\,db}{\sup_{b\in B}\int_{\pi_0}^{1-\pi_0} Q(b,q,r)^2\,dr} \tag{5.9}$$

Proof. Define $J(a) = a(1-a)$. Notice

(i) $k^{-4}(1-2\pi_0)^{-1}\sum_{n=\pi_0 k}^{(1-\pi_0)k} S^2_{m,m+[rk]}/J(n/k)$

$\to \sigma_u^{*2}\int_{\pi_0}^{1-\pi_0} Q(b,q,r)^2/J(r)\,dr$

(ii) $(T-k)^{-1}k^{-4}(1-2\pi_0)^{-1}\sum_{m=1}^{T-k}\sum_{n=\pi_0 k}^{(1-\pi_0)k} S^2_{m,m+[rk]}/J(n/k)$

$\rightarrow \sigma_u^{*2}\int_0^{1-q}\int_{\pi_0}^{1-\pi_0} Q(b,q,r)^2/J(r)\,dr\,db$

(iii) $k^{-4}(1-2\pi_0)^{-1}\sup_{m\in M}\sum_{n=\pi_0 k}^{(1-\pi_0)k} S^2_{m,m+[rk]}/J(n/k)$

$\rightarrow \sigma_u^{*2}\sup_{b\in B}\int_{\pi_0}^{1-\pi_0} Q(b,q,r)^2/J(r)\,dr$

follow from the application of Lemma 2. The result follows directly. Observe that $\int_{\pi_0}^{1-\pi_0} J(r)^{-1}dr$ factors out and cancels. □

Proposition 8 (Λ-**SupLM**(π_0)).

$$\Lambda_5(q) \longrightarrow \frac{\int_0^{1-q}\sup_{r\in[\pi_0,(1-\pi_0)]} Q(b,q,r)^2/J(r)\,db}{\sup_{b\in B}\sup_{r\in[\pi_0,(1-\pi_0)]} Q(b,q,r)^2/J(r)} \tag{5.10}$$

Proof. Define $J(a)=a(1-a)$. Notice

(i) $(T-k)^{-1}k^{-3}\sum_{m=1}^{T-k}\sup_{n\in[\pi_0 k,(1-\pi_0)k]} S^2_{m,m+[rk]}/J(n/k)$

$\rightarrow \sigma_u^{*2}\int_0^{1-q}\sup_{r\in[\pi_0,(1-\pi_0)]} Q(b,q,r)^2/J(r)\,dr$

(ii) $k^{-3}\sup_{m\in M}\sup_{n\in[\pi_0 k,(1-\pi_0)k]} S^2_{m,m+[rk]}/J(n/k)$

$\rightarrow \sigma_u^{*2}\sup_{b\in B}\sup_{r\in[\pi_0,(1-\pi_0)]} Q(b,q,r)^2/J(r)$

follow from the application of Lemma 2. The result follows directly. □

5.4 Critical Values

The asymptotic critical values for each $\Lambda_i(q)$ under the Stage II null hypothesis of a unit root were estimated by direct simulation for $q=\frac{1}{8}$ using samples of size $T=4000$ with 10,000 replications. The simulations were performed using the GAUSS matrix programming language. Table 5.1 presents the critical values for $\alpha = 0.01, 0.05, 0.10$ and 0.25. The choice of the subsample window, q, was driven by simulation results that show it to have good properties in discriminating against the alternative of a one-time, uniformly distributed structural break in commonly encountered sample sizes.

5.5 Normalized Break Sensitivity (NBS)

In our discussion of power, we focus on the Stage II alternative H_1. For that subset of H_a, we are able to provide analytic results by considering how sensitive the underlying Λ_i statistic is to a one-time structural change at π. To compare the sensitivity between the different Λ_i, it is helpful to look at the value of Λ_i given a pure break process with one break at time πT. Define a pure break process, y_t, with one break at time πT, $\pi \in [0, 1]$, of size δ, as

$$y_t(\pi) = \delta d_t(\pi), \qquad d_t(\pi) = \begin{cases} 1 & t \leq \pi T \\ 0 & t > \pi T \end{cases} \tag{5.11}$$

Consider the function $\eta(\pi)$ that maps $y_t(\pi)$ to its statistic value $\Lambda(y_t(\pi))$.

$$\eta(\pi) = \Lambda(y_t(\pi)) \tag{5.12}$$

$$\eta^*(\pi) = \frac{\eta(\pi)}{\sup\limits_{r \in [0,1]} \Lambda(y_t(r))} \tag{5.13}$$

We will refer to $\eta^*(\pi)$ as the Normalized Break Sensitivity (NBS) function. Denote by $\bar{\eta}$ and $\bar{\eta}^*$ the averages over all possible breaks π for (5.12) and (5.13) respectively.

To derive the NBS functions for several stability statistics of interest, it will prove convenient to find the partial sums and variance of $y_t(\pi)$. Define the residual $e_t = y_t(\pi) - \bar{y}(\pi)$ and the partial sum process $S_t = \sum_{i=1}^{t} e_i$. It follows that

$$S_t^2 = \begin{cases} (T\delta \frac{t}{T}(1-\pi))^2 & t \leq \pi T \\ (T\delta(1-\frac{t}{T})\pi)^2 & t > \pi T \end{cases} \tag{5.14}$$

$$\sigma^2 = \delta^2 \pi(1-\pi)$$

The following subsections derive $\eta_i(\pi)$ and compute the mean NBS $\bar{\eta}_i$ for each of the stability statistics of interest. These values will be important in showing the consistency of Λ_i as well as for the comparison of the empirical performance of the different underlying sub-sample statistics.

5.5.1 AHBT(r)

Consider the general AHBT(r) where $r = \frac{t}{T}$ is the point in the sample to test for a structural break when the true break is at point πT. It follows from (5.14) that

$$T^{-1}AHBT(r) = \frac{T^{-2}S_{rT}^2}{\sigma^2 r(1-r)} = \begin{cases} \frac{r(1-\pi)}{\pi(1-r)} & r \le \pi \\ \frac{\pi(1-r)}{r(1-\pi)} & r > \pi \end{cases} \tag{5.15}$$

5.5.2 AHBT(1/2)

From (5.15) it follows that

$$\eta_1(\pi) = \begin{cases} \frac{\pi}{(1-\pi)} & \pi \le \frac{1}{2} \\ \frac{(1-\pi)}{\pi} & \pi > \frac{1}{2} \end{cases}$$

Notice that $\sup \eta_1(\pi) = 1$ which obtains at the break-point $\pi = \frac{1}{2}$.

$$\begin{aligned} \eta_1 = \int_0^1 \eta_1(\pi)\, d\pi &= \int_0^{1/2} \frac{\pi}{\pi(1-\pi)}\, d\pi + \int_{1/2}^1 \frac{\pi(1-\pi)}{\pi}\, d\pi \\ &= 2\left(-\frac{1}{2} + \ln(2)\right) \approx 0.386 \end{aligned}$$

Since $\sup \eta_1(\pi) = 1$ at $\pi = 1/2$, the mean NBS is $\eta_1^* \approx 0.386$.

5.5.3 AvgS

$$\begin{aligned} T^{-1}AvgS &= T^{-3}\sum_{t=1}^{\pi T} S_t^2/\sigma^2 + T^{-3}\sum_{t=\pi T+1}^{T} S_t^2/\sigma^2 \\ &\to \int_0^{\pi} \frac{r^2(1-\pi)}{\pi}\, dr + \int_{\pi}^1 \frac{\pi(1-r)^2}{(1-\pi)}\, dr \end{aligned}$$

as $T \to \infty$. If we denote $\eta_2(\pi) = T^{-1}AvgS$ then

$$\begin{aligned} \eta_2 &= \int_0^1 \eta_2(\pi)\, d\pi \\ &= \int_0^1 \int_0^{\pi} \frac{r^2(1-\pi)}{\pi}\, dr\, d\pi + \int_0^1 \int_{\pi}^1 \frac{\pi(1-r)^2}{(1-\pi)}\, dr\, d\pi \\ &= 1/18 \end{aligned}$$

Since $\sup \eta_2(\pi) = 1/12$ at $\pi = 1/2$, the mean NBS is $\eta_2^* = 2/3$.

5.5.4 SupS

$$\begin{aligned} T^{-1}SupS &= \max_{t\in[1,T]} T^{-2}S_t^2/\sigma^2 \\ &\to \begin{cases} \sup\limits_{r\in[0,\pi]} \frac{r^2(1-\pi)}{\pi} \\ \sup\limits_{r\in[\pi,1]} \frac{\pi(1-r)^2}{(1-\pi)} \end{cases} \\ &= \pi(1-\pi) \end{aligned}$$

by recognition of the fact that the sup will occur at $r = \pi$ as $T \to \infty$. Denoting $\eta_3(\pi) = T^{-1}SupS = \pi(1-\pi)$ we have

$$\eta_3 = \int_0^1 \pi(1-\pi)\, d\pi = 1/6$$

Since $\sup \eta_3(\pi) = 1/4$ at $\pi = 1/2$, the mean NBS is $\eta_3^* = 2/3$.

5.5.5 AvgLM(0.10)

For $T \to \infty$

$$\begin{aligned} T^{-1}AvgLM(\pi_0) &= T^{-1}\sum_{t=\pi_0 T}^{\pi T} S_t^2/(\sigma^2 r(1-r)) + T^{-1}\sum_{t=\pi T+1}^{T(1-\pi_0)} S_t^2/(\sigma^2 r(1-r)) \\ &\to \int_{\pi_0}^{\pi} \frac{r(1-\pi)}{\pi(1-r)}\, dr + \int_{\pi}^{1-\pi_0} \frac{\pi(1-r)}{r(1-\pi)}\, dr \end{aligned}$$

If we denote $\eta_4(\pi) = T^{-1}AvgLM(\pi_0)$ then

$$\begin{aligned} \eta_2 &= \int_0^1 \eta_4(\pi)\, d\pi \\ &= \int_0^1\int_{\pi_0}^{\pi} \frac{r(1-\pi)}{\pi(1-r)}\, dr\, d\pi + \int_0^1\int_{\pi}^{1-\pi_0} \frac{\pi(1-r)}{r(1-\pi)}\, dr\, d\pi \\ &\approx 0.05411 \qquad \text{for } \pi_0 = 0.10 \end{aligned}$$

Since $\sup \eta_4(\pi) \approx 0.08267$ at $\pi = 1/2$, the mean NBS is $\eta_4^* = 0.655$.

5.5.6 SupLM(0.10)

The NBS function for SupLM(π_0) can be divided into three parts: $\pi \in [0, \pi_0]$, $\pi \in [\pi_0, 1-\pi_0]$, and $\pi \in [1-\pi_0, 1]$. Notice that because of the restriction, breaks in the ends of the sample will result in a test result identical to that of AHBT(π_0) and AHBT($1-\pi_0$), respectively.

For breaks between π_0 and $1-\pi_0$

$$
\begin{aligned}
T^{-1}SupLM(\pi_0) &= \max_{t \in [\pi_0 T, T(1-\pi_0)T]} T^{-2}S_t^2/(\sigma^2 r(1-r)) \\
&\rightarrow \begin{cases} \sup\limits_{r\in[0,\pi]} \frac{r(1-\pi)}{\pi(1-r)} \\ \sup\limits_{r\in[\pi,1]} \frac{\pi(1-r)}{r(1-\pi)} \end{cases} \\
&= 1
\end{aligned}
$$

by recognition of the fact that the sup will occur at $r = \pi$ as $T \rightarrow \infty$. Denoting $\eta_5(\pi) = T^{-1}SupLM(\pi_0)$ we have

$$
\begin{aligned}
\eta_5^* &= \int_0^{\pi_0} \frac{\pi(1-\pi_0)}{\pi_0(1-\pi)}\,d\pi + \int_{\pi_0}^{1-\pi_0} 1\,d\pi + \int_{1-\pi_0}^{1} \frac{\pi_0(1-\pi)}{\pi(1-\pi_0)}\,d\pi \\
&\approx 0.0482 + 0.8 + 0.0482 = 0.8964 \qquad \text{for } \pi_0 = 0.10
\end{aligned}
$$

5.6 Consistency and Power

We are now ready to consider the convergence of Λ under H_1, a one-time structural break uniformly distributed over the sample interval. For any given series with one break, the sub-sample statistic Λ can be written as the sum of two parts. The first term, denoted $\dot{\Lambda}$, consists of those sub-samples that overlap the break to some extent while the second term, denoted Λ, consists all those sub-samples that do not.

$$
\Lambda = \dot{\Lambda} + \Lambda. \tag{5.16}
$$

where

$$\dot{\Lambda} = (T-k)^{-1} \sum_{m=1}^{T-k} \frac{I(m,k)\Lambda(m,k)}{\sup_{j\in[1,T-k]} \Lambda(j,k)}$$

$$\Lambda = (T-k)^{-1} \sum_{m=1}^{T-k} \frac{(1-I(m,k))\Lambda(m,k)}{\sup_{j\in[1,T-k]} \Lambda(j,k)}$$

and T is the sample size, k is the sub-sample length, and $I(m,k)$ is an indicator function that takes the value of 1 if there is a break in the sub-sample $\{y_m, \ldots, y_{m+k}\}$.

If the break is uniformly distributed within the sample period, then $E(\Lambda)$ can be written as

$$E(\Lambda) = pE(\Lambda^{mid}) + (1-p)E(\Lambda^{end}) \tag{5.17}$$

Here, p is the probability that the break is in the middle (break points between [k,T-k], $k < 1/2$) of the sample and the superscripts *mid* and *end* refer to the value of the statistic when the break is in the middle and ends ($[1,k-1]$ and $[T-k+1,T]$) of the sample, respectively. $E(\Lambda^{mid})$ is the same for all breaks in the middle. Breaks that occur in the ends of the sample are slightly different because there are fewer samples that overlap the break and more that do not.

These two decompositions of Λ help to set up the following proposition which allows us to differentiate between a one-time, uniformly distributed structural break and a unit root.

Proposition 9. *Let $\eta_i^* = \int_0^1 \eta_i^*(\pi)\,d\pi$, be the average NBS function for a sub-sample statistic, Λ_i. Fix $q = k/T$ at a constant value between 0 and 1/2. Under H_1, as $T \to \infty$,*

(i) $\Lambda_i^{mid}(q) \to \frac{1}{(1-q)}\eta_i^*$ *For* $\pi \in [k, T-k]$.

(ii) $\Lambda_i^{end}(q) \le \frac{1}{(1-q)}\eta_i^*$ *For* $\pi \le k/T$ *or* $\pi \ge 1-k/T$.

Proof. Given a structural break at time πT, the model $y_t = \delta d_t(\pi) + \varepsilon_t$ where $d_t(\pi)$ is given by (5.11). Equation (5.16) yields

$$\Lambda_i^{mid}(k/T) = \frac{(T-k)^{-1}}{\sup_{j\in[1,T-k]} \Lambda_i(j,k)} \sum_{m=1}^{T-k} \Lambda_i(m,k)$$

$$= \frac{(T-k)^{-1}}{\sup_{j\in[1,T-k]} \Lambda_i(j,k)} \left[\sum_{m=1}^{\pi T-k-1} \Lambda_i(m,k) + \sum_{m=\pi T-k}^{\pi T} \Lambda_i(m,k) + \sum_{m=\pi T+1}^{T-k} \Lambda_i(m,k) \right]$$

This follows from the observation that given a subsample of length k, a break in the middle of the sample, i.e. between the points k and $T-k$, will result in k sub-samples that overlap the break-point and $T-2k$ sub-samples that do not. The k sub-samples that overlap the break will yield $\eta^*(m)$, which is the NBS function from Section 5.5. The remaining $T-2k$ sub-samples that do not overlap the break will be stationary series.

$$\begin{aligned}
\Lambda_t^{mid}(k/T) &= \frac{(T-k)^{-1}}{\sup\limits_{j\in[1,T-k]} \lambda_t(j,k)}\left[o_p(1) + \sum_{m=\pi T-k}^{\pi T} \lambda_t(m,k) + o_p(1)\right] \\
&= \frac{(T-k)^{-1}}{\sup\limits_{j\in[1,T-k]} \lambda_t(j,k)}\left[\sum_{m=\pi T-k}^{\pi T} \eta_t\left(1-\frac{m-(\pi T-k)}{k}\right) + o_p(1)\right] \\
&= (T-k)^{-1}\sum_{m=\pi T-k}^{\pi T} \eta_t^*\left(1-\frac{m-(\pi T-k)}{k}\right)
\end{aligned}$$

Then as $T\to\infty$, $q=\frac{k}{T}$ constant

$$\Lambda_t^{mid}(q) \longrightarrow \frac{q}{1-q}\eta_t^*.$$

which proves (i).

To prove (ii), notice that if we define π^{mid} to be a break in the middle of the sample

$$\sum_{j=\pi^{mid}T-k}^{\pi^{mid}T} \eta(j) = c, \qquad \forall\ \pi^{mid}\in[k,T-k].$$

In particular, for $\pi^{mid}=k/T$ we have $\sum_{j=1}^{k}\eta(j)=c$. Then, for breaks in the ends, $\pi_1^{end}\le k/T$ and $\pi_2^{end}\ge(1-k/T)$, we have

$$\begin{aligned}
\sum_{j=1}^{\pi_1^{end}T}\eta(j) &\le \sum_{j=1}^{k}\eta(j) \\
\sum_{j=\pi_2^{end}T}^{T}\eta(j) &\le \sum_{j=T-k}^{T}\eta(j).
\end{aligned}$$

Then (ii) follows directly. □

Proposition 9 shows that the power of Λ against a uniformly distributed structural break will only depend upon the distribution of the statistic for those breaks that occur in the middle of the sample. Further, it shows that the distribution of Λ for a break in the middle converges to a constant that depends upon the sub-sample statistic λ_t through its NBS function and the sub-sample length q. It also provides a way to compare the effect on power of using different underlying stability statistics λ_t in the computation of the sub-sample test Λ_t. Given two statistics, the one with the largest difference between its critical value under H_∞ and η_t^* will have more power under H_1.

The unconditional expectation of $\Lambda(q)$ under a uniformly distributed structural break has a surprisingly simple form that highlights the basis for the proof of consistency in Proposition 9. It is given by the following result.

Proposition 10. *Let $\eta_t^* = \int_0^1 \eta_t^*(\pi)\,d\pi$, be the average NBS function for a sub-sample statistic, Λ_t. Let $q = k/T$ be fixed at a constant value between 0 and 1/2. Under H_1,*

$$E(\Lambda_t(q)) \rightarrow q\eta_t^* \quad \text{as } T \rightarrow \infty.$$

Proof. From Proposition 9 we have $\Lambda_t^{mid}(q) \longrightarrow \frac{q}{1-q}\eta_t^*$. Breaks at the ends of the sample are slightly different. By using the symmetry of the beginning and end of the sample combined with the symmetry of $\eta^*(i)$, we can match each break point $\pi T < k$ with its complimentary break at point $T - k + \pi T$. More formally, $\sum_{i=m}^{m+k} \eta(i) = \sum_{i=1}^{\pi T} \eta(i) + \sum_{i=T-k+\pi T}^{T} \eta(i)$. Thus, by combining two series of size T we get k synthetic overlapping sub-samples with full $\eta^*(i)$ functions that are analogous to those where the break occurs in the middle of the sample. These k overlapping sub-samples are matched with $2T - 3k$ non-overlapping sub-samples. By averaging we get an expected $k/2$ overlapping and $T - \frac{3}{2}k$ non-overlapping sub-samples for series that have a break in one of the ends of the sample. It follows that as $T \rightarrow \infty$, $q = \frac{k}{T}$ constant

$$E(\Lambda_t^{end}(q)) = \frac{q}{2(1-q)}\eta_t^*$$

Putting these two results together, we can find the unconditional expectation. As $T \rightarrow \infty$, $q = \frac{k}{T}$ constant

$$\begin{aligned} E(\Lambda_t(q)) &\rightarrow (1-2q)\left(\frac{q}{1-q}\right)\eta_t^* + 2q\left(\frac{q}{2(1-q)}\right)\eta_t^* && (5.18) \\ &= (1-q)^{-1}\left((q-2q^2)\eta_t^* + q^2\eta_t^*\right) \\ &= q\eta_t^* \end{aligned}$$

□

5.7 Empirical Properties

To evaluate our two-stage subsample test, we divide the outcomes into three categories. For a given simulation, H_0 gives the proportion of outcomes that fail to reject at Stage I. If there is a rejection at Stage I, we record the Stage II results as H_1 or $H_{\sim}$ depending on whether or not the Stage II outcome rejects the unit root hypothesis. Every simulation must end up in one of these three categories.

5.7.1 Simple Λ Tests

In this exercise, the columns of Table 5.2–5.7 contain the percentage of outcomes that end up in each category. Therefore, a lower percentage in H_0 will imply higher Stage I power. For alternatives where a Stage I rejection is likely, our goal is to have the Stage II outcome concentrated in the correct category, H_1 or $H_{\sim}$, for the given alternative.

Panels A–D of Tables 5.2–5.4 show size of Λ for the primal statistics AHBT, AvgS, SupS, SupLM($\pi_0 = 0.10$) under a Unit Root. All four tests have good size properties. The AHBT has difficulty rejecting H_0 in Stage I and its power suffers, even in the larger samples compared to the other tests.

Panels A–D of Tables 5.5–5.7 present the power of Λ for the four aforementioned test statistics against the alternative of one uniformly distributed structural break. The H_0 Stage I rejections match the power of the primal statistics from the tables in Chapter 3. Overall the power of the test increases with both break size and sample size. Interestingly, the AHBT beats the other tests across the board for H_1 vs. $H_{\sim}$ but has trouble with H_0. The Stage II tests work well when the Stage I null, H_0, is rejected with high frequency.

Panels A–D of Tables 5.8–5.13 look at the power for two other interesting alternatives in H_n, a break in the interior of the sample and a fractionally integrated process. The tests perform slightly worse for the interior break as for a single uniformly distributed break. Again the AHBT looks to perform the best. By contrast, the $I(d)$ process is almost always classified as a Unit Root, particularly for the non-stationary values where $d \geq 1/2$.

5.7.2 Hybrid Sub-sample Tests

The empirical results for the simple subsample test lead us to propose a hybrid subsample test that improves upon the usefulness of the Λ test by utilizing a different statistic in each stage of the test. Panels E–G of Tables 5.2–5.4 show the size of a Stage II AHBT combined with Stage I AvgS, SupS, and SupLM respectively. The hybrid tests significantly improve the size properties of the AHBT.

Panels E–G of Tables 5.5–5.13 present the power of the hybrid tests. The hybrid tests are the more powerful primarily because they avoid the weakness of the AHBT which is getting past Stage I.

5.8 Summary

If you evaluate differences in terms of testing outcomes, then there is very little practical difference between structural change and other forms of nonstationarity such as unit roots and fractional integration. This chapter addresses this issue by investigating the properties of a test designed to discriminate between a unit root and a process with one structural break that is uniformly distributed within the sample period. The test applies common tests of stability to sub-samples of the data and compares the average of these sub-sample statistics to their maximum within the sub-samples. The distribution of this Stage II statistic is derived under the null hypothesis of a unit root and the size and power is shown to perform well across choices of the underlying stability statistic.

The best test is shown to be a hybrid of the statistics for the Stage I and Stage II tests. Using the AvgS, SupS, or SupLM statistics for the Stage I test and then the AHBT for the Stage II test overcomes the size and power problems of the AHBT for H_0 while retaining its sensitivity in differentiating H_1 from $H_\sim$. Among these tests, the hybrid test that uses the SupLM for the Stage I test seems to be the most powerful.

Table 5.1: Critical Values for $\Lambda(q)$

	1%	5%	10%	25%
AHBT	0.096	0.118	0.131	0.159
AvgS	0.109	0.132	0.148	0.176
SupS	0.123	0.148	0.164	0.197
AvgLM $_{10}$	0.111	0.134	0.149	0.178
SupLM $_{10}$	0.134	0.162	0.179	0.214

Table 5.2: Unit Root DGP: A(1/8), 5% cv, T=200

ahbt

sigma	H0	H1	Hinf
0.000	0.946	0.016	0.038
0.100	0.269	0.133	0.598
0.200	0.172	0.124	0.704
0.500	0.110	0.076	0.814
1.000	0.094	0.053	0.853
2.000	0.097	0.060	0.843
5.000	0.089	0.054	0.857

avgS

sigma	H0	H1	Hinf
0.000	0.953	0.001	0.046
0.100	0.147	0.023	0.830
0.200	0.042	0.037	0.921
0.500	0.003	0.045	0.952
1.000	0.001	0.038	0.961
2.000	0.000	0.042	0.958
5.000	0.000	0.047	0.953

supS

sigma	H0	H1	Hinf
0.000	0.955	0.001	0.044
0.100	0.138	0.003	0.859
0.200	0.032	0.017	0.951
0.500	0.004	0.034	0.962
1.000	0.000	0.040	0.960
2.000	0.000	0.052	0.948
5.000	0.000	0.062	0.938

supLM

sigma	H0	H1	Hinf
0.000	0.962	0.000	0.038
0.100	0.124	0.002	0.874
0.200	0.022	0.017	0.961
0.500	0.002	0.040	0.958
1.000	0.000	0.051	0.949
2.000	0.000	0.068	0.932
5.000	0.000	0.081	0.919

ahbt2(H0:avgS)

sigma	H0	H1	Hinf
0.000	0.953	0.011	0.036
0.100	0.147	0.151	0.702
0.200	0.042	0.148	0.810
0.500	0.003	0.088	0.909
1.000	0.001	0.062	0.937
2.000	0.000	0.062	0.938
5.000	0.000	0.061	0.939

ahbt3(H0:supS)

sigma	H0	H1	Hinf
0.000	0.955	0.009	0.036
0.100	0.138	0.156	0.706
0.200	0.032	0.153	0.815
0.500	0.004	0.088	0.908
1.000	0.000	0.062	0.938
2.000	0.000	0.062	0.938
5.000	0.000	0.061	0.939

ahbt4(H0:supLM)

sigma	H0	H1	Hinf
0.000	0.962	0.010	0.028
0.100	0.124	0.153	0.723
0.200	0.022	0.153	0.825
0.500	0.002	0.089	0.909
1.000	0.000	0.062	0.938
2.000	0.000	0.062	0.938
5.000	0.000	0.061	0.939

Table 5.3: Unit Root DGP: A(1/8), 5% cv, T=500

ahbt

sigma	H0	H1	Hinf
0.000	0.946	0.020	0.034
0.100	0.116	0.141	0.743
0.200	0.095	0.063	0.842
0.500	0.063	0.063	0.874
1.000	0.057	0.060	0.883
2.000	0.055	0.057	0.888
5.000	0.055	0.069	0.876

avgS

sigma	H0	H1	Hinf
0.000	0.938	0.000	0.062
0.100	0.016	0.056	0.928
0.200	0.000	0.045	0.955
0.500	0.000	0.046	0.954
1.000	0.000	0.048	0.952
2.000	0.000	0.051	0.949
5.000	0.000	0.050	0.950

supS

sigma	H0	H1	Hinf
0.000	0.936	0.000	0.064
0.100	0.012	0.019	0.969
0.200	0.000	0.028	0.972
0.500	0.000	0.050	0.950
1.000	0.000	0.061	0.939
2.000	0.000	0.053	0.947
5.000	0.000	0.064	0.936

supLM

sigma	H0	H1	Hinf
0.000	0.948	0.000	0.052
0.100	0.008	0.014	0.978
0.200	0.000	0.035	0.965
0.500	0.000	0.064	0.936
1.000	0.000	0.075	0.925
2.000	0.000	0.070	0.930
5.000	0.000	0.080	0.920

ahbt2(H0:avgS)

sigma	H0	H1	Hinf
0.000	0.938	0.023	0.039
0.100	0.016	0.155	0.829
0.200	0.000	0.076	0.924
0.500	0.000	0.070	0.930
1.000	0.000	0.063	0.937
2.000	0.000	0.059	0.941
5.000	0.000	0.072	0.928

ahbt3(H0:supS)

sigma	H0	H1	Hinf
0.000	0.936	0.028	0.036
0.100	0.012	0.156	0.832
0.200	0.000	0.076	0.924
0.500	0.000	0.070	0.930
1.000	0.000	0.063	0.937
2.000	0.000	0.059	0.941
5.000	0.000	0.072	0.928

ahbt4(H0:supLM)

sigma	H0	H1	Hinf
0.000	0.948	0.028	0.024
0.100	0.008	0.157	0.835
0.200	0.000	0.076	0.924
0.500	0.000	0.070	0.930
1.000	0.000	0.063	0.937
2.000	0.000	0.059	0.941
5.000	0.000	0.072	0.928

Table 5.4: Unit Root DGP: A(1/8), 5% cv, T=1000

ahbt

sigma	H0	H1	Hinf
0.000	0.948	0.018	0.034
0.100	0.077	0.100	0.823
0.200	0.070	0.056	0.874
0.500	0.051	0.052	0.897
1.000	0.036	0.047	0.917
2.000	0.034	0.052	0.914
5.000	0.046	0.059	0.895

avgS

sigma	H0	H1	Hinf
0.000	0.948	0.000	0.052
0.100	0.000	0.060	0.940
0.200	0.000	0.044	0.956
0.500	0.000	0.041	0.959
1.000	0.000	0.046	0.954
2.000	0.000	0.042	0.958
5.000	0.000	0.045	0.955

supS

sigma	H0	H1	Hinf
0.000	0.948	0.000	0.052
0.100	0.000	0.036	0.964
0.200	0.000	0.042	0.958
0.500	0.000	0.051	0.949
1.000	0.000	0.047	0.953
2.000	0.000	0.052	0.948
5.000	0.000	0.058	0.942

supLM

sigma	H0	H1	Hinf
0.000	0.952	0.000	0.048
0.100	0.000	0.046	0.954
0.200	0.000	0.054	0.946
0.500	0.000	0.061	0.939
1.000	0.000	0.058	0.942
2.000	0.000	0.071	0.929
5.000	0.000	0.075	0.925

ahbt2(H0:avgS)

sigma	H0	H1	Hinf
0.000	0.948	0.015	0.037
0.100	0.000	0.107	0.893
0.200	0.000	0.062	0.938
0.500	0.000	0.053	0.947
1.000	0.000	0.048	0.952
2.000	0.000	0.055	0.945
5.000	0.000	0.061	0.939

ahbt3(H0:supS)

sigma	H0	H1	Hinf
0.000	0.948	0.015	0.037
0.100	0.000	0.107	0.893
0.200	0.000	0.062	0.938
0.500	0.000	0.053	0.947
1.000	0.000	0.048	0.952
2.000	0.000	0.055	0.945
5.000	0.000	0.061	0.939

ahbt4(H0:supLM)

sigma	H0	H1	Hinf
0.000	0.952	0.018	0.030
0.100	0.000	0.107	0.893
0.200	0.000	0.062	0.938
0.500	0.000	0.053	0.947
1.000	0.000	0.048	0.952
2.000	0.000	0.055	0.945
5.000	0.000	0.061	0.939

Table 5.5: Uniformly Distributed Str. Brk. DGP: A(1/8), 5% cv, T=200

ahbt

brkSize	H0	H1	Hinf
0.000	0.953	0.011	0.036
0.100	0.939	0.017	0.044
0.200	0.867	0.027	0.106
0.500	0.535	0.115	0.350
1.000	0.272	0.326	0.402
2.000	0.147	0.788	0.065
5.000	0.088	0.912	0.000

avgS

brkSize	H0	H1	Hinf
0.000	0.950	0.000	0.050
0.100	0.930	0.001	0.069
0.200	0.856	0.000	0.144
0.500	0.435	0.005	0.560
1.000	0.166	0.088	0.746
2.000	0.074	0.636	0.290
5.000	0.033	0.964	0.003

supS

brkSize	H0	H1	Hinf
0.000	0.953	0.000	0.047
0.100	0.939	0.000	0.061
0.200	0.869	0.000	0.131
0.500	0.447	0.001	0.552
1.000	0.163	0.038	0.799
2.000	0.072	0.522	0.406
5.000	0.035	0.964	0.001

supLM

brkSize	H0	H1	Hinf
0.000	0.960	0.000	0.040
0.100	0.957	0.000	0.043
0.200	0.885	0.000	0.115
0.500	0.440	0.001	0.559
1.000	0.126	0.015	0.859
2.000	0.052	0.401	0.547
5.000	0.025	0.974	0.001

ahbt2(H0:avgS)

brkSize	H0	H1	Hinf
0.000	0.950	0.013	0.037
0.100	0.930	0.015	0.055
0.200	0.856	0.029	0.115
0.500	0.435	0.145	0.420
1.000	0.166	0.379	0.455
2.000	0.074	0.851	0.075
5.000	0.033	0.962	0.005

ahbt3(H0:supS)

brkSize	H0	H1	Hinf
0.000	0.953	0.014	0.033
0.100	0.939	0.013	0.048
0.200	0.869	0.030	0.101
0.500	0.447	0.145	0.408
1.000	0.163	0.383	0.454
2.000	0.072	0.855	0.073
5.000	0.035	0.963	0.002

ahbt4(H0:supLM)

brkSize	H0	H1	Hinf
0.000	0.960	0.013	0.027
0.100	0.957	0.009	0.034
0.200	0.885	0.028	0.087
0.500	0.440	0.144	0.416
1.000	0.126	0.401	0.473
2.000	0.052	0.866	0.082
5.000	0.025	0.967	0.008

Table 5.6: Uniformly Distributed Str. Brk. DGP: A(1/8), 5% cv, T=500

ahbt

brkSize	H0	H1	Hinf
0.000	0.964	0.013	0.023
0.100	0.891	0.038	0.071
0.200	0.742	0.098	0.160
0.500	0.351	0.277	0.372
1.000	0.176	0.686	0.138
2.000	0.098	0.898	0.004
5.000	0.040	0.958	0.002

avgS

brkSize	H0	H1	Hinf
0.000	0.950	0.001	0.049
0.100	0.883	0.001	0.116
0.200	0.669	0.009	0.322
0.500	0.229	0.056	0.715
1.000	0.108	0.423	0.469
2.000	0.059	0.929	0.012
5.000	0.017	0.982	0.001

supS

brkSize	H0	H1	Hinf
0.000	0.960	0.000	0.040
0.100	0.877	0.000	0.123
0.200	0.682	0.001	0.317
0.500	0.218	0.007	0.775
1.000	0.108	0.199	0.693
2.000	0.057	0.928	0.015
5.000	0.016	0.983	0.001

supLM

brkSize	H0	H1	Hinf
0.000	0.952	0.000	0.048
0.100	0.894	0.000	0.106
0.200	0.714	0.001	0.285
0.500	0.173	0.008	0.819
1.000	0.086	0.140	0.774
2.000	0.045	0.928	0.027
5.000	0.013	0.987	0.000

ahbt2(H0:avgS)

brkSize	H0	H1	Hinf
0.000	0.950	0.020	0.030
0.100	0.883	0.044	0.073
0.200	0.669	0.124	0.207
0.500	0.229	0.338	0.433
1.000	0.108	0.744	0.148
2.000	0.059	0.935	0.006
5.000	0.017	0.979	0.004

ahbt3(H0:supS)

brkSize	H0	H1	Hinf
0.000	0.960	0.020	0.020
0.100	0.877	0.048	0.075
0.200	0.682	0.119	0.199
0.500	0.218	0.341	0.441
1.000	0.108	0.742	0.150
2.000	0.057	0.938	0.005
5.000	0.016	0.980	0.004

ahbt4(H0:supLM)

brkSize	H0	H1	Hinf
0.000	0.952	0.023	0.025
0.100	0.894	0.035	0.071
0.200	0.714	0.110	0.176
0.500	0.173	0.357	0.470
1.000	0.086	0.756	0.158
2.000	0.045	0.944	0.011
5.000	0.013	0.981	0.006

Table 5.7: Uniformly Distributed Str. Brk. DGP: A(1/8), 5% cv, T=1000

ahbt

brkSize	H0	H1	Hinf
0.000	0.955	0.019	0.026
0.100	0.851	0.070	0.079
0.200	0.608	0.169	0.223
0.500	0.241	0.475	0.284
1.000	0.121	0.866	0.013
2.000	0.045	0.951	0.004
5.000	0.022	0.976	0.002

avgS

brkSize	H0	H1	Hinf
0.000	0.952	0.001	0.047
0.100	0.823	0.007	0.170
0.200	0.514	0.011	0.475
0.500	0.146	0.176	0.678
1.000	0.071	0.790	0.139
2.000	0.025	0.973	0.002
5.000	0.010	0.990	0.000

supS

brkSize	H0	H1	Hinf
0.000	0.955	0.001	0.044
0.100	0.820	0.000	0.180
0.200	0.504	0.002	0.494
0.500	0.138	0.037	0.825
1.000	0.070	0.631	0.299
2.000	0.025	0.972	0.003
5.000	0.010	0.990	0.000

supLM

brkSize	H0	H1	Hinf
0.000	0.959	0.000	0.041
0.100	0.832	0.001	0.167
0.200	0.516	0.001	0.483
0.500	0.112	0.022	0.866
1.000	0.053	0.542	0.405
2.000	0.018	0.979	0.003
5.000	0.008	0.991	0.001

ahbt2(H0:avgS)

brkSize	H0	H1	Hinf
0.000	0.952	0.023	0.025
0.100	0.823	0.071	0.106
0.200	0.514	0.204	0.282
0.500	0.146	0.539	0.315
1.000	0.071	0.913	0.016
2.000	0.025	0.970	0.005
5.000	0.010	0.989	0.001

ahbt3(H0:supS)

brkSize	H0	H1	Hinf
0.000	0.955	0.021	0.024
0.100	0.820	0.074	0.106
0.200	0.504	0.213	0.283
0.500	0.138	0.548	0.314
1.000	0.070	0.914	0.016
2.000	0.025	0.972	0.003
5.000	0.010	0.989	0.001

ahbt4(H0:supLM)

brkSize	H0	H1	Hinf
0.000	0.959	0.019	0.022
0.100	0.832	0.076	0.092
0.200	0.516	0.211	0.273
0.500	0.112	0.563	0.325
1.000	0.053	0.924	0.023
2.000	0.018	0.977	0.005
5.000	0.008	0.991	0.001

Table 5.8: Uniformly Distributed Interior Brk: $\lambda(1/8)$, 5% cv, T=200

ahbt

brkSize	H0	H1	Hinf
0.000	0.947	0.008	0.045
0.100	0.946	0.010	0.044
0.200	0.888	0.030	0.082
0.500	0.710	0.068	0.222
1.000	0.498	0.209	0.293
2.000	0.300	0.574	0.126
5.000	0.185	0.813	0.002

avgS

brkSize	H0	H1	Hinf
0.000	0.947	0.000	0.053
0.100	0.950	0.000	0.050
0.200	0.900	0.003	0.097
0.500	0.644	0.009	0.347
1.000	0.294	0.061	0.645
2.000	0.123	0.297	0.580
5.000	0.059	0.455	0.486

supS

brkSize	H0	H1	Hinf
0.000	0.956	0.000	0.044
0.100	0.948	0.000	0.052
0.200	0.910	0.000	0.090
0.500	0.620	0.000	0.380
1.000	0.286	0.015	0.699
2.000	0.110	0.212	0.678
5.000	0.052	0.624	0.324

supLM

brkSize	H0	H1	Hinf
0.000	0.963	0.000	0.037
0.100	0.960	0.000	0.040
0.200	0.923	0.000	0.077
0.500	0.658	0.000	0.342
1.000	0.276	0.005	0.719
2.000	0.125	0.095	0.780
5.000	0.056	0.388	0.556

ahbt2(H0:avgS)

brkSize	H0	H1	Hinf
0.000	0.947	0.012	0.041
0.100	0.950	0.007	0.043
0.200	0.900	0.023	0.077
0.500	0.644	0.078	0.278
1.000	0.294	0.286	0.420
2.000	0.123	0.722	0.155
5.000	0.059	0.939	0.002

ahbt3(H0:supS)

brkSize	H0	H1	Hinf
0.000	0.956	0.009	0.035
0.100	0.948	0.011	0.041
0.200	0.910	0.024	0.066
0.500	0.620	0.094	0.286
1.000	0.286	0.296	0.418
2.000	0.110	0.734	0.156
5.000	0.052	0.947	0.001

ahbt4(H0:supLM)

brkSize	H0	H1	Hinf
0.000	0.963	0.008	0.029
0.100	0.960	0.012	0.028
0.200	0.923	0.023	0.054
0.500	0.658	0.083	0.259
1.000	0.276	0.301	0.423
2.000	0.125	0.722	0.153
5.000	0.056	0.943	0.001

Table 5.9: Uniformly Distributed Interior Brk: Λ(1/8), 5% cv, T=500

ahbt

brkSize	H0	H1	Hinf
0.000	0.936	0.019	0.045
0.100	0.926	0.024	0.050
0.200	0.858	0.047	0.095
0.500	0.543	0.164	0.293
1.000	0.339	0.492	0.169
2.000	0.169	0.813	0.018
5.000	0.128	0.871	0.001

avgS

brkSize	H0	H1	Hinf
0.000	0.940	0.000	0.060
0.100	0.933	0.001	0.066
0.200	0.857	0.005	0.138
0.500	0.388	0.049	0.563
1.000	0.167	0.253	0.580
2.000	0.066	0.537	0.397
5.000	0.042	0.427	0.531

supS

brkSize	H0	H1	Hinf
0.000	0.942	0.001	0.057
0.100	0.926	0.000	0.074
0.200	0.837	0.000	0.163
0.500	0.358	0.004	0.638
1.000	0.149	0.078	0.773
2.000	0.057	0.480	0.463
5.000	0.035	0.657	0.308

supLM

brkSize	H0	H1	Hinf
0.000	0.944	0.000	0.056
0.100	0.935	0.000	0.065
0.200	0.852	0.000	0.148
0.500	0.369	0.003	0.628
1.000	0.142	0.033	0.825
2.000	0.059	0.336	0.605
5.000	0.037	0.415	0.548

ahbt2(H0:avgS)

brkSize	H0	H1	Hinf
0.000	0.940	0.016	0.044
0.100	0.933	0.025	0.042
0.200	0.857	0.049	0.094
0.500	0.388	0.237	0.375
1.000	0.167	0.622	0.211
2.000	0.066	0.912	0.022
5.000	0.042	0.957	0.001

ahbt3(H0:supS)

brkSize	H0	H1	Hinf
0.000	0.942	0.016	0.042
0.100	0.926	0.029	0.045
0.200	0.837	0.055	0.108
0.500	0.358	0.248	0.394
1.000	0.149	0.639	0.212
2.000	0.057	0.920	0.023
5.000	0.035	0.963	0.002

ahbt4(H0:supLM)

brkSize	H0	H1	Hinf
0.000	0.944	0.021	0.035
0.100	0.935	0.019	0.046
0.200	0.852	0.048	0.100
0.500	0.369	0.248	0.383
1.000	0.142	0.642	0.216
2.000	0.059	0.919	0.022
5.000	0.037	0.962	0.001

Table 5.10: Uniformly Distributed Interior Brk: Λ(1/8), 5% cv, T=1000

ahbt

brkSize	H0	H1	Hinf
0.000	0.946	0.020	0.034
0.100	0.907	0.035	0.058
0.200	0.749	0.090	0.161
0.500	0.414	0.334	0.252
1.000	0.240	0.697	0.063
2.000	0.133	0.865	0.002
5.000	0.084	0.916	0.000

avgS

brkSize	H0	H1	Hinf
0.000	0.959	0.001	0.040
0.100	0.912	0.004	0.084
0.200	0.704	0.008	0.288
0.500	0.238	0.136	0.626
1.000	0.109	0.429	0.462
2.000	0.055	0.513	0.432
5.000	0.022	0.437	0.541

supS

brkSize	H0	H1	Hinf
0.000	0.949	0.000	0.051
0.100	0.907	0.000	0.093
0.200	0.685	0.001	0.314
0.500	0.212	0.018	0.770
1.000	0.090	0.265	0.645
2.000	0.045	0.619	0.336
5.000	0.016	0.729	0.255

supLM

brkSize	H0	H1	Hinf
0.000	0.955	0.000	0.045
0.100	0.922	0.000	0.078
0.200	0.699	0.000	0.301
0.500	0.217	0.004	0.779
1.000	0.096	0.133	0.771
2.000	0.042	0.426	0.532
5.000	0.021	0.435	0.544

ahbt2(H0:avgS)

brkSize	H0	H1	Hinf
0.000	0.959	0.017	0.024
0.100	0.912	0.031	0.057
0.200	0.704	0.105	0.191
0.500	0.238	0.447	0.315
1.000	0.109	0.819	0.072
2.000	0.055	0.944	0.001
5.000	0.022	0.977	0.001

ahbt3(H0:supS)

brkSize	H0	H1	Hinf
0.000	0.949	0.020	0.031
0.100	0.907	0.032	0.061
0.200	0.685	0.115	0.200
0.500	0.212	0.464	0.324
1.000	0.090	0.838	0.072
2.000	0.045	0.953	0.002
5.000	0.016	0.983	0.001

ahbt4(H0:supLM)

brkSize	H0	H1	Hinf
0.000	0.955	0.019	0.026
0.100	0.922	0.029	0.049
0.200	0.699	0.108	0.193
0.500	0.217	0.462	0.321
1.000	0.096	0.834	0.070
2.000	0.042	0.955	0.003
5.000	0.021	0.978	0.001

Table 5.11: I(d) DGP: A(1/8), 5% cv. T=200

ahbt

d	H0	H1	Hinf
0.000	0.964	0.002	0.034
0.250	0.600	0.045	0.355
0.500	0.291	0.077	0.632
0.750	0.160	0.065	0.775
1.000	0.093	0.061	0.846

avgS

d	H0	H1	Hinf
0.000	0.957	0.000	0.043
0.250	0.455	0.009	0.536
0.500	0.066	0.035	0.899
0.750	0.006	0.045	0.949
1.000	0.000	0.051	0.949

supS

d	H0	H1	Hinf
0.000	0.959	0.000	0.041
0.250	0.400	0.001	0.599
0.500	0.051	0.016	0.933
0.750	0.004	0.038	0.958
1.000	0.001	0.062	0.937

supLM

d	H0	H1	Hinf
0.000	0.962	0.000	0.038
0.250	0.362	0.001	0.637
0.500	0.029	0.012	0.959
0.750	0.002	0.031	0.967
1.000	0.000	0.059	0.941

ahbt2(H0:avgS)

d	H0	H1	Hinf
0.000	0.957	0.008	0.035
0.250	0.455	0.071	0.474
0.500	0.066	0.094	0.840
0.750	0.006	0.076	0.918
1.000	0.000	0.062	0.938

ahbt3(H0:supS)

d	H0	H1	Hinf
0.000	0.959	0.009	0.032
0.250	0.400	0.075	0.525
0.500	0.051	0.095	0.854
0.750	0.004	0.076	0.920
1.000	0.001	0.062	0.937

ahbt4(H0:supLM)

d	H0	H1	Hinf
0.000	0.962	0.012	0.026
0.250	0.362	0.087	0.551
0.500	0.029	0.097	0.874
0.750	0.002	0.076	0.922
1.000	0.000	0.062	0.938

Table 5.12: I(d) DGP: A(1/8), 5% cv, T=500

ahbt

d	H0	H1	Hinf
0.000	0.938	0.018	0.044
0.250	0.482	0.085	0.433
0.500	0.202	0.099	0.699
0.750	0.102	0.056	0.842
1.000	0.057	0.039	0.904

avgS

d	H0	H1	Hinf
0.000	0.943	0.000	0.057
0.250	0.239	0.023	0.738
0.500	0.009	0.034	0.957
0.750	0.000	0.042	0.958
1.000	0.000	0.036	0.964

supS

d	H0	H1	Hinf
0.000	0.945	0.000	0.055
0.250	0.163	0.002	0.835
0.500	0.003	0.013	0.984
0.750	0.000	0.028	0.972
1.000	0.000	0.036	0.964

supLM

d	H0	H1	Hinf
0.000	0.959	0.000	0.041
0.250	0.151	0.003	0.846
0.500	0.000	0.006	0.994
0.750	0.000	0.025	0.975
1.000	0.000	0.037	0.963

ahbt2(H0:avgS)

d	H0	H1	Hinf
0.000	0.943	0.020	0.037
0.250	0.239	0.126	0.635
0.500	0.009	0.116	0.875
0.750	0.000	0.060	0.940
1.000	0.000	0.041	0.959

ahbt3(H0:supS)

d	H0	H1	Hinf
0.000	0.945	0.019	0.036
0.250	0.163	0.142	0.695
0.500	0.003	0.116	0.881
0.750	0.000	0.060	0.940
1.000	0.000	0.041	0.959

ahbt4(H0:supLM)

d	H0	H1	Hinf
0.000	0.959	0.019	0.022
0.250	0.151	0.146	0.703
0.500	0.000	0.116	0.884
0.750	0.000	0.060	0.940
1.000	0.000	0.041	0.959

Table 5.13: I(d) DGP: A(1/8), 5% cv, T=1000

ahbt

d	H0	H1	Hinf
0.000	0.936	0.022	0.042
0.250	0.435	0.111	0.454
0.500	0.121	0.098	0.781
0.750	0.069	0.065	0.866
1.000	0.044	0.041	0.915

avgS

d	H0	H1	Hinf
0.000	0.942	0.001	0.057
0.250	0.158	0.036	0.806
0.500	0.001	0.040	0.959
0.750	0.000	0.054	0.946
1.000	0.000	0.042	0.958

supS

d	H0	H1	Hinf
0.000	0.948	0.001	0.051
0.250	0.088	0.004	0.908
0.500	0.000	0.014	0.986
0.750	0.000	0.037	0.963
1.000	0.000	0.038	0.962

supLM

d	H0	H1	Hinf
0.000	0.951	0.000	0.049
0.250	0.075	0.002	0.923
0.500	0.000	0.008	0.992
0.750	0.000	0.032	0.968
1.000	0.000	0.044	0.956

ahbt2(H0:avgS)

d	H0	H1	Hinf
0.000	0.942	0.019	0.039
0.250	0.158	0.152	0.690
0.500	0.001	0.106	0.893
0.750	0.000	0.072	0.928
1.000	0.000	0.043	0.957

ahbt3(H0:supS)

d	H0	H1	Hinf
0.000	0.948	0.018	0.034
0.250	0.088	0.164	0.748
0.500	0.000	0.106	0.894
0.750	0.000	0.072	0.928
1.000	0.000	0.043	0.957

ahbt4(H0:supLM)

d	H0	H1	Hinf
0.000	0.951	0.015	0.034
0.250	0.075	0.165	0.760
0.500	0.000	0.106	0.894
0.750	0.000	0.072	0.928
1.000	0.000	0.043	0.957

Chapter 6

Fixed-Income Returns and Yields: Long Memory or Structural Instability?

Unlike equity returns, many fixed-income return measures appear to display considerable long memory. This holds particularly strongly for shorter-maturity Treasury securities in the U.S. In a number of recent papers, Granger has argued that long memory in returns may only reflect infrequent structural breaks. He finds the case for long memory in volatility much stronger. Parke (1999) develops a model that generates long memory through a type of structural shift. In this paper, we show that the extent of long memory depends crucially on whether gross or excess returns are under consideration and we provide a simple demonstration of why this distinction is so important. We also explore the impact of structural instability on tests for long memory using a version of the *supLM* test developed by Andrews (1993). Briefly, we find evidence of long memory in gross returns, yields and term-premia even after accounting for structural shifts in a number of different ways.

6.1 Introduction

Finance research on the Treasury security market is extensive, and there are many papers that characterize different features of the distribution of Treasury security returns and volatility. This paper extends the existing research by documenting the long memory properties of Treasury security returns and by presenting some evidence on the sources of the long memory. Our research is prompted by the fact that bill and bond returns display a leading characteristic of long memory: their autocorrelations are large, but die out very slowly (especially compared to equity return autocorrelations). This autocorrelation property is an important feature of fractionally-differenced time-series models, analyzed by Granger (1980), Hosking (1981), and Geweke and Porter-Hudak (1983).

Perhaps the most important question to address is why finance researchers should take an interest in long-memory models, particularly since applications of these models to equity data have found little credible evidence of long memory. One direct answer is that unlike the equity literature, we find strong evidence of long memory in all of our data series except excess returns. This empirical regularity is important because Mandelbrot (1971) shows that there may be arbitrage opportunities in asset markets with long memory.

Financial risk management systems typically use time-series representations of return behavior, but long memory does not appear to be incorporated into these products.[1] This assumption may be a reasonable approximation for short-horizon risk management, but neglected long-memory components in return and volatility phenomena may lead to inaccuracies in modeling and managing longer-horizon risks. The consequences of using an inappropriate time series model in this setting are not well known, but probably merit study for some, longer-horizon risk management problems.

The unique long-horizon forecasting properties of long-memory models (which we discuss in Section 6.2) make them interesting to study, especially given the current interest in return predictability, particularly at long horizons. Andersson (1998) shows that ignoring long memory in forecasting exercises when it exists is worse than imposing long memory when it does not exist. Long memory is also important for pricing models. Backus and Zin (1993), Bollerslev and Mikkelsen (1996), and Comte and Renault (1996) are a few examples of papers which explore the consequences of long memory for pricing bonds, equity options, and interest rates options.

We document the long memory properties of Treasury Bill and Bond gross and excess holding period returns, yields and the term-premium. To do this, we test for long memory using a test statistic developed by Kwiatkowski, Phillips, Schmidt, and Shin (1992). We show

[1] Riskmetrics is one example. The documentation provided for the software appears to indicate clearly that ARIMA(p, d, q) models with d set either to zero or to one are standard in this risk management product.

that weekly gross holding period returns on Treasury Bills display strong evidence of long memory, even after accounting for short-term dependence in the series. Treasury Bond holding period returns, on the other hand, do not appear to have long memory. Interestingly, we find that excess returns on longer-maturity bills and bonds show no evidence of long memory. We also find a strong degree of persistence in yields and term-premium across securities.

What produces long memory? Recent work makes it clear that structural instability may produce (spurious) evidence of long memory. Lobato and Savin (1998) warn empirical investigators about the possibility that structural instability may lead to misinterpretation of long memory evidence. Granger and Hyung (1999) show that a linear process with structural breaks can mimic long memory series and present simulation evidence that long memory in absolute S&P 500 returns is more likely due to structural breaks than an underlying I(d) process. Hightower and Parke (1999) demonstrate that certain structural stability tests and particular tests for long memory are related to one another: each is a specific function of a common statistic based on the cumulative sums of the error process. This implies there is ambiguity in the interpretation of tests for long memory: evidence of an I(d) process may actually be structural instability in disguise. For these reasons and because U.S. debt markets have experienced change over our sample period, we report a detailed analysis of how particular sample partitions (corresponding to specific market changes) affect the evidence for long memory. We also use the *supLM* test of Andrews (1993) and the sequential break test of Bai and Perron (2001) to identify likely structural breaks in the time series.

Our empirical work uses a sample of hand-collected weekly holding period returns on seven (nearly constant-maturity) Treasury bills and bonds covering the July 1962 May 1996 period. Section 6.3 of the paper reports further details on the sample

In the next section, we discuss the properties of fractionally-differenced time series, their potential use in modeling expected returns, and a test for long memory. Section 6.3 describes our data. Section 6.4 reports our empirical results for the long-term memory test and analyzes the structural stability issues A final section summarizes the issues considered in the paper and discusses the implications of our findings.

6.2 Long-Memory Processes

6.2.1 Introduction to Long Memory

Normally, only integer powers of d are considered in ARIMA(p, d, q) models, but there is no mathematical or statistical requirement that d take on only integer values (e.g., $d=1$ yields a first-difference model). In a fractionally-differenced model, d can take on non-integer values and the resulting time series can exhibit some particularly interesting dependencies. Granger and Joyeux (1980) and Hosking (1981) show that extending the lag operator to non-integer powers of d results in a well-defined time series that is fractionally integrated of order d.[2] The differencing operator may be written

$$(1-L)^d = \sum_{k=0}^{\infty}(-1)^k \binom{d}{k} L^k \qquad (6.1)$$

leading to the following representation of a time series where $p = q = 0$:

$$(1-L)^d y_t = \sum_{k=0}^{\infty} \frac{\Gamma(k-d)}{\Gamma(-d)\Gamma(k+1)} y_{t-k} \qquad (6.2)$$

Here, Γ is the usual gamma function.

In his excellent survey paper, Baillie (1996) reviews a number of different long-memory models. One simple model is an ARFIMA $(0, d, 0)$ process given by

$$(1-L)^d (y_t - \mu) = \epsilon_t \qquad (6.3)$$

This model is studied in Granger (1980), Granger and Joyeux (1980), and Hosking (1981). Their work shows than when $d < .5$, the series has finite variance, but for $d = .5$, the series has infinite variance. The time series is stationary and invertible when $-.5 < d < .5$. For $d = .5$, standard Box-Jenkins techniques will indicate that differencing is required and provided that $d < 1$, differencing will produce a series whose spectrum is zero at zero frequency. This heavily-used model is a special case of an ARFIMA (p, d, q) process given by

$$\Phi(L)(1-L)^d (y_t - \mu) = \Theta(L)\epsilon_t \qquad (6.4)$$

where $p = q = 0$.

[2] See also Robinson (1978) for early analysis of long-memory models.

Fractionally-differenced time-series models have very interesting long-run forecasting properties. A fractional white noise series $y_t \sim I(d)$ may be represented as an $MA(\infty)$ process where the moving average coefficients decline slowly following the form $b_j \sim A_j^{d-1}$ where A is a constant. A stationary $ARMA(p,q)$ with infinite p and q will have coefficients that decline at least exponentially: $b_j \sim A\theta^j$. One important implication of these stark differences in coefficient decay rates is that a fractionally-differenced model may provide better long-run forecasts from a very simple model compared to $ARMA(p,q)$ models where p and q are large.[3]

In principal, parameterizations of both finite-order ARMA and fractionally-differenced time series can produce dependence in a time series. The rate at which past information ceases to be useful in forecasting future values differs importantly across these models. A comparison of the autocorrelograms for a fractionally-differenced time series with d=0.4 and an AR(1) process with ρ=0.5 provide a nice illustration. The first autocorrelation for each series is nearly identical (0.5 vs. 0.6) but the decay rates are quite different. The autocorrelations decline very slowly for the fractionally-differenced series, but fall quite rapidly to zero for the AR(1) process. This is an example of why fractionally-differenced time series display greater persistence than AR (or ARMA) processes, and why they may be interesting in research on debt instrument yield and return distributions.[4]

6.2.2 Models of Persistent Expected Returns

Existing time series models of expected returns can be cast in terms of ARFIMA models. To see this, begin with the following expected return model:

$$R_t = E_{t-1}(R_t) + \epsilon_t \qquad (6.5)$$

where R_t is the asset return at time t, E is the expectations operator, and ϵ_t is a mean zero, constant variance error term with the property $E(\epsilon_t, \epsilon_{t-j}) = 0$ for all j. If the expected return

[3] For further discussion of the autocorrelation, autocovariance, and general forecasting properties of long-memory models, see Baillie (1996).

[4] In fact, Lo (1991) shows that a fractionally-differenced model can reproduce the general pattern of variance ratio results reported in the equity literature. In particular, he shows that a combination of an AR(1) and fractionally-differenced model with d=0.25 will produce variance ratios above one at short horizons and below one at longer horizons. This suggests that fractionally-differenced models may have special importance in ongoing empirical investigations of long-range dependence in capital markets. Lo's own results suggest that there is little evidence of long-term memory in U.S. equity index returns once short-run dependencies have been accounted for in tests for long-memory.

is a constant (μ), we may write

$$R_t - \mu = \epsilon_t \tag{6.6}$$

This, together with assumptions about the properties of the ϵ_t series, yields the common random walk model of asset prices, a special case of ARFIMA where p=0, d=1, and q=0.

More generally, varying assumptions about the expected return process can produce particular special cases of an ARIMA(p, d, q) model whose general form is

$$\Phi(L)(1 - L)^d(R_t) = \Theta(L)\epsilon_t \tag{6.7}$$

where all terms are as defined in the previous section. Kan and Lee (1991) show that the constant expected return model obtains when d=0 in (6.5). Hence, finding a nonzero value of d implies the presence of long-memory components in asset returns. This implies lagged returns will be useful in forecasting long-horizon future returns (i.e., there is persistence in asset returns).

6.2.3 Testing for Long Memory

Kwiatkowski, Phillips, Schmidt, and Shin (1992) develop a test for I(0) behavior which is consistent against an I(d) alternative and can be helpful in distinguishing long memory from short memory. The null hypothesis of their test is that a time series is I(0), but under the alternative hypothesis, the time series displays I(d) behavior (with $d < 1$). Lee and Schmidt (1996) provide further analysis of this approach to testing for long-memory effects. Their Monte Carlo evidence suggests that the KPSS test has power comparable to Lo's robust R/S statistic in distinguishing I(0) from I(d) behavior.

The first step in calculating the KPSS test statistic is to form the partial sum (S_t) of the residuals from the demeaned series.[5] The test statistic is given by

$$\hat{\eta}_\mu = T^{-2} \sum_{j=1}^{n} S_j^2 / s_T^2(\ell) \tag{6.8}$$

where the denominator is the autocorrelation-consistent variance estimator defined by

$$s^2(\ell) = T^{-1} \sum_{t=1}^{T} \hat{\epsilon}_t^2 + 2T^{-1} \sum_{s=1}^{T} w(s, \ell) \sum_{t=s+1}^{T} \hat{\epsilon}_t \hat{\epsilon}_{t-s} \tag{6.9}$$

[5] There is another version of their test, η_τ, which is constructed in the same way except that the residuals are derived from a regression that involves a time trend as well as intercept term.

This robust variance estimator is based on Phillips (1987), who demonstrates its consistency under certain conditions and Newey and West (1987), who suggested the weighting scheme $w(s,\ell) = 1 - j/(\ell + 1)$ to guarantee that the variance estimate is positive semi-definite.

Lo (1991) shows that short-range dependence (well documented in equity prices by Lo and MacKinlay (1988), and Conrad and Kaul (1989)) may compromise inferences about the presence of long-range dependence. Embedded in the KPSS test is a somewhat different view of long-range dependence than is apparent in the long-horizon equity returns literature. Aside from some technical conditions, the null hypothesis of no long-range dependence eliminates infinite variance marginal distributions and encompasses a strong-mixing condition that requires higher-order autocorrelations to fall in size as the lag length increases. This means that the series of autocorrelations displayed by a time series under the null hypothesis decays rapidly. Included in the null hypothesis, then, are all finite-order ARMA models. The null hypothesis of no long-range dependence includes well-known models of return dependence (see Campbell et al. (1997) for details). Put another way, the null hypothesis in the KPSS test excludes return behavior that is quite different from the autocorrelation series commonly reported in earlier work.

Since the optimal number of autocovariances is not known ex ante, we compute $\hat{\eta}_{m}u$ using a number of different autocovariances, ℓ. The tradeoff is that using too few autocovariances produces an inadequate bias correction, but using too many leads to low power since higher-order autocovariances are more imprecisely estimated.

6.3 Data

In the subsequent empirical analysis, we analyze weekly *gross and excess holding period returns* on U.S. Treasury securities. The basic return data set contains weekly holding period returns on one-, three-, six-, and 12-month Treasury Bills and three-, five-, and 10-year Treasury Bonds for the July 1962 through May 1996 period.[6]

Weekly holding period returns were calculated by taking the log difference of this Wednesday's bid price and last Wednesday's ask price and adding in the percentage return associated

[6] The original weekly return data was collected by Gautam Kaul and very graciously provided to us. We are grateful to Paisan Limratanamongkol who updated all the data series for us.

with accrued interest. The bid and ask prices used to calculate the weekly return were for the same security. However, the security used in the computations was frequently changed so as to maintain a fairly constant maturity return series. In no case, though, were prices on different securities used to make return computations. The basic Treasury bill and bond price data is from the *Wall Street Journal.*[7]

We computed six excess return measures: the weekly holding period return on 10-year bonds (and five-year., three-year., 12-month, six-month, and three-month) less the weekly holding period return on one-month bonds. This gives the extra return earned by holding a longer-maturity Treasury debt instrument vs. short-maturity Treasury debt.

6.4 Empirical Results

6.4.1 Full Sample Results

We use the KPSS statistic to test for long memory. The results of the KPSS test on the full sample returns, excess returns, yields, and term-premia are reported in the panels of Table 6.1 labeled "Full Sample".

For all of the return series except the weekly returns on three-, five- and 10-year Treasury bonds, the null hypothesis of stationarity is strongly rejected in favor of an $I(d)$ process. The KPSS test statistics clearly imply that weekly gross fixed income holding period returns display long memory. The evidence of persistence is even stronger in the yields and term-premia, where almost all of the series reject for all choices of autocorrelation truncation parameter, ℓ. The exception is the 12-month over 3-month term-premium, which fails to reject stationarity when $\ell < 0$.

For weekly *excess* holding period returns, the KPSS values show very little evidence of long-term dependence. This stands in sharp contrast to our just-noted findings about long memory in gross holding period returns. It suggests that empirical asset pricing work that focuses on excess returns can safely ignore the implications of neglected long memory.

[7] One motivation for using individual security data is to avoid aggregation of multiple security returns where possible. Granger (1980) provides an analysis of conditions under which aggregation can produce long memory.

6.4.2 Sample Selection and Structural Stability

While our time series samples of holding period returns are fairly long, we need to address a fundamental issue in all empirical studies of long-memory processes. With a stable underlying structure, high frequency effects may be found by sampling very frequently, but over a relatively short time period (i.e., sample every 15 seconds for three business days). The long-memory phenomenon we are interested in may be measured accurately only in long samples, i.e., samples which extend over many realizations of the long-memory process. This is perhaps best achieved with samples that cover many years (say, 200 years), but where the process is not sampled with high frequency. What is needed for long- memory empirical studies is a long sample realization of the process (i.e., 200 years), particularly one that is not sampled so often that short-run dependencies dominate the sample properties of the data.

The difficulty here is that the underlying structures of debt markets, instruments, and trading institutions and practices have not been stable over periods of even 50 years, let alone 100 to 200 years.[8] As Lobato and Savin (1998) suggest, this instability may lead to spurious evidence of long memory.[9] In our setting, the task is to provide the longest possible sample while recognizing that extending the length of the sample increases the probability of structural instability.[10]

Lobato and Savin (1998) assess the fragility of evidence for long memory by splitting their sample of daily returns and squared returns into sub-samples. They recompute their tests for, and measures of, long memory for each sub-sample, and then comparing inferences from the whole sample and the sub-samples.[11] The analysis in Hightower and Parke (1999) indicates that structural shift tests and the KPSS test for $I(d)$ (versus $I(0)$) behavior are both functions of the same term: the ratio of the partial sums of the series to a consistent variance estimate for

[8] One example is the Treasury Fed Accord of 1953 that ended the Fed's explicit policy of managing Treasury borrowing costs. A more recent example is the shift in Fed operating procedures in October 1979.

[9] Indeed, Diebold (1986) argues that evidence of integrated in variance GARCH, a nonstationary conditional variance model, is really an indication of underlying instability. Lastrapes (1989) provides considerable evidence in favor of Diebold's interpretation.

[10] Ideally, instability tests such as those developed in Hansen (1992) might be used to resolve the instability issue empirically. Unfortunately, these tests are not available for our application. Hidalgo and Robinson (1996) have studied the issue of structural change in the mean with long memory for the case where the time series are Gaussian. There is substantial evidence rejecting Gaussianity for financial market return series, so this test does not seem to be particularly appropriate for the problem we are studying here.

[11] There is evidence of sub-sample instability from the equity market mean-reversion literature. In particular, see Kim and Nelson (1998) and the references therein.

that series. One interpretation of their theoretical results is that "large" KPSS test statistics can be interpreted as evidence of a structural shift. This implies it is particularly important to assess the robustness of the results reported for the full samples to changes in the sample endpoints.

We explore the fragility of long memory evidence by comparing inferences across sub-samples and our full sample. As a crude check on our earlier findings, we recalculated the KPSS test statistics using these sample breakpoints and checked the stability of the long-range dependence test statistics across the early and later samples. In this way, we hope to provide some initial evidence on the impact of potential structural instability on our inferences about long memory.

6.4.3 First-Pass Stability Analysis

As a first-pass stability analysis we split our sample into two parts. More specifically, because of the shift in Federal Reserve System operating procedures in early October 1979, we split our U.S. Treasury bill and bond data at September 1979. The various panels of Tables 6.1–6.4 contain full sample and sub-sample estimates of the KPSS test statistics for all the returns for which we reported earlier. For each group, (returns, excess returns, yields, and term-premium) the first panel is the KPSS test statistic for the full sample period. The second (third) panel is the KPSS test statistic for the first (second) sub-sample.

Analysis of the U.S. Treasury sub-sample results reveals some very interesting regularities. First, the evidence for long memory from the KPSS tests is pervasive in returns, yields, and term-premia across the two sub-samples. The only exceptions are longer maturity returns in the full and pre-October 1979 period and the 10-year bond for full and sub-samples. Second, there is very little evidence of persistence in excess returns.[12] These findings are robust over a wide choice of bandwidth choices to account for short-run dynamics.

[12] We found very similar behavior in the monthly return data. Our results on excess returns were quite robust across the sub-samples. We find no evidence of long memory in excess returns in either sub-sample.

6.4.4 Finding Break Points with SupLM Tests

The relationship between long memory and structural change has received an increasing amount or attention in recent years. Some researchers (Diebold and Inoue (2001) and Granger and Hyung (1999)) see long memory as an artifact of processes that exhibit certain types of structural change over time. Others (Parke (1999) and Taqqu, Willinger, and Sherman (1997)) propose models where many structural breaks that last for random durations give rise to time series properties that are associated with long memory (slowly decaying autocorrelations). Taken together, these lines of research point to a blurring of the differences between long memory and structural change.

In empirical studies, whether a process is deemed to be long memory or a break may come down to what types of tests are performed on the data. Hightower and Parke (1999) clarify this point by showing that the commonly performed KPSS test for stationarity, which has also shown to be a consistent test against long memory (Lee and Schmidt), is an algebraic special case of the Andrews and Ploberger (1994) tests for structural change. As these tests have nearly identical empirical properties, the modeling choice between long memory and structural change has typically been determined by the test run.

Andrews (1993) suggests a supremum test for a one-time structural change with an unknown break point as a way to account for the criticism that researchers may "eyeball" the most likely point for a break before running a typical test. Given a time series y_t, residuals defined in the usual manner as $e_t = y_t - \bar{y}$, and defining $S_t = \sum_{i=1}^{t} e_i$ to be the cumulative sum of residuals, the LM test for a one-time change in mean at a point $\pi \in (0, 1)$ can be shown to be

$$LM_T(\pi) \cong T^{-1} \frac{S_{[\pi T]}^2}{\pi(1-\pi)s^2(\ell)} \qquad (6.10)$$

where $s^2(\ell)$ is given by equation (6.9). If $w(s, \ell)$ is taken to be the Bartlett kernel, $s^2(\ell)$ is identical to the denominator of the KPSS statistic. The Andrews $supLM$ test is then simply given by $\sup_{\pi \in \Pi} LM_T(\pi)$ where Π is bounded away from 0 and 1.

The relationship between tests of stationarity and tests of structural change can be seen by a comparison of (6.8) and (6.10). The KPSS test for stationarity can be viewed as an average of the $LM_T(\pi)$'s weighted by $\pi(1-\pi)$. Hightower and Parke (1999) show that these two tests (as well as the Andrews and Ploberger $avgLM$ and $expLM$ tests) have nearly identical power against many common alternatives including structural change, unit roots and long memory. In this context, the $supLM$ test can provide insight into candidate breakpoints that may be driving rejections of short memory, if indeed these rejections are driven by a structural shift in the sample.

This idea has been expanded on by Bai and Perron (2001) who show that multiple breakpoints may be estimated by using a sequential procedure. We use their procedure to see if instability in monetary variables and inflation can explain the persistence seen above.

6.4.5 Splitting Using M2

One idea is that instability in the treasury debt-market securities is driven by instability in an underlying monetary aggregate, such as M2. To test this theory, we sequentially estimate break points in M2 for our sample period using the Bai-Perron procedure. We then use these break points to split our return and yield (as well as our excess return and term-premium) series and retest the resultant subsamples using the KPSS test. The idea is that if the instability/persistence is coming from monetary variables, the evidence for long memory will be significantly decreased in the debt securities once we take in into account.

The results are shown in Tables 6.5–6.8. We identify four sub-samples in M2 and test within them accordingly. The conclusions from the naive split remain unchanged except, perhaps in the second subsample. However, this sub-sample is quite short (78 observations) so it is wise not to put too much weight on this reversal.

6.4.6 Splitting Using Inflation

Next, we consider some of the facts from our initial exercise. We find persistence in returns but not excess returns as well as in yields and term-premium. In the term-premium, the evidence increases with the maturity differential. One explanation for these facts could be an underlying instability in inflation driving the persistence in the debt-market securities through the Fisher equation. If nominal rates are equal to the real rate plus expected inflation, we would expect to see just this type of behavior.[13]

[13] Recall that expected inflation is over the life of the bond and therefore different for bonds of differing maturity

Tables 6.9–6.12 presents the results from sequentially splitting samples using inflation. The procedure picks a three-break model with breaks at 1967:5, 1973:1, and 1982:7. These are shown in Figure 6.3. This selection of break dates closely agrees with those of Bai and Perron (2001) for the U.S. real interest rate (they use quarterly a 3-month Treasury bill deflated by the CPI).

Once again, this split seems to have very little effect on the persistence in the series. An interesting feature of this experiment is that persistence increases for long-bonds in the fourth sub-sample, suggesting an increase in instability or long memory since the 1980's. The persistence is also confirmed by the results of a test Busetti and Harvey (2001) propose to test for persistence in the presence of multiple structural breaks. The results of this test, which we denote CvM for Cramer-von Mises, are shown in Table 6.5. Notice that the test does not reject stationarity for inflation, confirming the results of the Bai-Perron procedure in estimating the break dates in inflation.

6.4.7 Markov-Switching Models

As a final effort to see if the persistence in treasury securities can be explained by underlying structural instability we estimate the state probabilities from a two-state Markov-switching model like that of Hamilton (1990). Figures 6.4 through 6.9 plot the state probabilities for a sample of our series. It seems that the Markov-switching model may provide an explanation of what we see in our tests for persistence. If evidence of long memory is taken to be equivalent to evidence of structural instability then the smoothed state probabilities for a two state switching model are quite interesting. For the series where we find evidence of long memory in our previous tables, there seems to be clear evidence of two regimes that switch over time. Further, the states cluster so that there are long runs of each state. On the other hand, for the series where we don't find much evidence of long memory (long bond returns and excess returns) the states switch back and forth frequently making the identification of runs of a specific regime difficult. In particular, notice that many of the subsamples chosen from the inflation break dates overlap regimes from the Markov-switching model.

Table 6.13 contains the results from splitting the samples on the major regime switches from the two-state Markov-switching model. On the surface they seem to present much the same picture as before. However, when viewed in combination with the state probability graphs it can be seen that the evidence of persistence (embodied in the magnitude of the statistics) is decreased and many previous strong rejections become marginal ones. This is a result of the difficulty of choosing "major" changes in regime. It appears that the small rejections of stability within a sub-sample are driven by very brief changes in regime. We calculated the CtM statistics using changes in regime as break dates. The results, presented in the second panel of Table 6.14, agree with the other attempts. The persistence in the series remains.

6.5 Summary and Further Discussion

One aim of this paper is to explore the utility of long memory models for understanding fixed-income market behavior. We can summarize our contributions in the answers to several questions.

The first question is whether fixed income returns have long memory properties similar to what has been documented for equity returns. Recall that evidence for long memory in gross equity returns is thin.[14] In this paper, we have shown that weekly, gross holding period returns on all but long bonds show significant evidence of long memory, particularly since October 1979. If we focus instead on weekly holding period returns in excess of the return on the shortest maturity bill, there is no compelling evidence for long memory in fixed income returns.

The second question is whether the evidence of long memory is due to underlying structural instability. We have gone to some lengths to establish the sensitivity of these findings to structural instability. There is no test available that permits us to unconditionally distinguish long memory from structural instability. Nonetheless, we have shown that many of our results on long memory hold in sub-samples. In this respect, we believe that our results have placed a significantly greater burden on researchers who would argue that fixed income research should ignore long memory because it is probably produced by structural instability. This issue is not yet settled, but as we indicated in the introduction to the paper, its resolution is important for a number of research areas in financial economics.

[14] Granger (1999) argues that there is no reason to expect long memory in return series, but he accepts long memory in return volatilities.

In many ways, our answers to these questions raise an even larger number of questions. We believe there are some interesting issues to address in exploring the implications of long memory for fixed income research. One step is to study the impact of long memory on term structure, bond pricing, and fixed income derivative models. Specifically, it would be useful to assess the size of any pricing biases and to identify the circumstances where the impact of ignoring long memory is the most (and least) noticeable. A second issue involves applying stochastic and deterministic models of long memory to yields and returns and comparing the forecasting performance of these models with existing, GARCH-related models. A third question revolves around the implications of long memory for longer-horizon, fixed-income risk management problems. Perhaps the most fundamental issue remains the economic foundations for long memory in fixed income markets.

Table 6.1: KPSS Test: Returns

Full Sample

ℓ	0	4	8	12
1-month	22.298	5.334	3.069	2.178
3-month	12.067	4.420	2.806	2.098
6-month	5.109	2.632	1.932	1.595
12-month	1.991	1.235	1.034	0.951
3-year	0.601	0.405	0.356	0.347
5-year	0.415	0.310	0.272	0.260
10-year	0.174	0.141	0.129	0.128

First Half of Sample (1–900)

ℓ	0	4	8	12
1-month	32.327	7.460	4.316	3.061
3-month	13.020	5.399	3.561	2.641
6-month	5.806	3.231	2.527	2.099
12-month	1.393	0.918	0.823	0.742
3-year	0.539	0.369	0.348	0.340
5-year	0.319	0.232	0.221	0.215
10-year	0.122	0.085	0.086	0.084

Second Half of Sample (901–1769)

ℓ	0	4	8	12
1-month	42.398	10.537	6.107	4.366
3-month	22.157	8.362	5.380	4.093
6-month	8.567	4.532	3.362	2.831
12-month	2.918	1.835	1.551	1.472
3-year	1.396	0.935	0.811	0.792
5-year	1.114	0.841	0.725	0.690
10-year	0.455	0.383	0.344	0.343

Table 6.2: KPSS Test: Excess Returns

Full Sample

ℓ	0	4	8	12
3m-1m	0.708	0.562	0.515	0.493
6m-1m	0.209	0.154	0.140	0.138
12m-1m	0.231	0.164	0.150	0.150
3y-1m	0.422	0.293	0.261	0.257
5y-1m	0.243	0.186	0.165	0.159
10y-1m	0.097	0.079	0.073	0.072

First Half of Sample (1 900)

ℓ	0	4	8	12
3m-1m	0.278	0.260	0.259	0.241
6m-1m	0.151	0.116	0.116	0.118
12m-1m	0.111	0.081	0.079	0.076
3y-1m	0.106	0.073	0.069	0.068
5y-1m	0.105	0.076	0.073	0.071
10y-1m	0.065	0.045	0.045	0.044

Second Half of Sample (901 1769)

ℓ	0	4	8	12
3m-1m	0.974	0.761	0.703	0.694
6m-1m	0.215	0.158	0.142	0.140
12m-1m	0.141	0.100	0.092	0.093
3y-1m	0.391	0.272	0.241	0.239
5y-1m	0.412	0.321	0.281	0.270
10y-1m	0.180	0.154	0.139	0.139

Table 6.3: KPSS Test: Yields

Full Sample

ℓ	0	4	8	12
3-month	30.114	6.078	3.412	2.388
6-month	30.950	6.240	3.501	2.449
12-month	33.886	6.829	3.828	2.676
3-year	45.847	9.226	5.163	3.602
5-year	52.849	10.628	5.941	4.140
10-year	63.024	12.661	7.068	4.917

First Half of Sample (1–900)

ℓ	0	4	8	12
3-month	39.557	8.062	4.557	3.206
6-month	42.799	8.705	4.916	3.459
12-month	47.825	9.722	5.491	3.865
3-year	64.515	13.073	7.358	5.158
5-year	71.096	14.374	8.070	5.643
10-year	78.760	15.887	8.897	6.206

Second Half of Sample (901–1769)

ℓ	0	4	8	12
3-month	58.795	11.892	6.697	4.703
6-month	60.778	12.277	6.906	4.844
12-month	62.365	12.587	7.073	4.955
3-year	64.880	13.076	7.333	5.128
5-year	65.780	13.254	7.428	5.190
10-year	67.247	13.542	7.581	5.290

Table 6.4: KPSS Test: Term Premium

Full Sample

ℓ	0	4	8	12
12m-1m	1.807	0.391	0.230	0.169
3y-1m	27.981	5.799	3.333	2.383
5y-1m	32.850	6.753	3.856	2.742
10y-1m	39.207	8.017	4.560	3.232

First Half of Sample (1–900)

ℓ	0	4	8	12
12m-1m	3.735	0.848	0.507	0.371
3y-1m	11.386	2.374	1.360	0.963
5y-1m	10.947	2.260	1.287	0.908
10y-1m	11.240	2.304	1.307	0.921

Second Half of Sample (901–1769)

ℓ	0	4	8	12
12m-1m	9.829	2.097	1.236	0.910
3y-1m	4.211	0.882	0.514	0.374
5y-1m	6.178	1.283	0.743	0.537
10y-1m	9.512	1.965	1.133	0.816

Table 6.5: KPSS Test on M2 Splits: Returns

First Break (1–344)

ℓ	0	4	8	12
1-month	20.548	5.164	3.041	2.187
3-month	14.315	4.760	2.955	2.162
6-month	4.953	2.481	1.795	1.453
12-month	1.461	0.919	0.741	0.639
3-year	0.106	0.070	0.060	0.053
5-year	0.102	0.070	0.059	0.054
10-year	0.060	0.045	0.049	0.050

Second Break (345–423)

ℓ	0	4	8	12
1-month	0.950	0.438	0.329	0.263
3-month	0.363	0.294	0.231	0.234
6-month	0.081	0.071	0.070	0.086
12-month	0.143	0.105	0.107	0.140
3-year	0.395	0.281	0.284	0.330
5-year	0.311	0.220	0.224	0.267
10-year	0.045	0.044	0.054	0.090

Third Break (424–1285)

ℓ	0	4	8	12
1-month	19.450	4.950	2.898	2.085
3-month	8.951	3.720	2.496	1.937
6-month	3.478	1.946	1.525	1.333
12-month	1.614	1.019	0.887	0.847
3-year	0.334	0.226	0.208	0.215
5-year	0.340	0.251	0.225	0.220
10-year	0.312	0.246	0.230	0.231

Fourth Break (1286–1769)

ℓ	0	4	8	12
1-month	14.518	3.612	2.063	1.419
3-month	11.113	4.173	2.474	1.763
6-month	4.487	2.640	1.785	1.358
12-month	1.159	0.915	0.726	0.636
3-year	0.304	0.248	0.211	0.200
5-year	0.150	0.136	0.124	0.123
10-year	0.116	0.106	0.094	0.093

Table 6.6: KPSS Test on M2 Splits: Excess Returns

First Break (1–344)

ℓ	0	4	8	12
3m-1m	1.327	1.041	0.915	0.801
6m-1m	0.306	0.222	0.195	0.186
12m-1m	0.146	0.107	0.094	0.087
3y-1m	0.158	0.108	0.092	0.082
5y-1m	0.264	0.182	0.155	0.141
10y-1m	0.148	0.112	0.120	0.123

Second Break (345–423)

ℓ	0	4	8	12
3m-1m	0.127	0.135	0.124	0.149
6m-1m	0.040	0.037	0.040	0.055
12m-1m	0.107	0.080	0.084	0.115
3y-1m	0.371	0.265	0.267	0.313
5y-1m	0.300	0.213	0.217	0.259
10y-1m	0.047	0.046	0.056	0.093

Third Break (424–1285)

ℓ	0	4	8	12
3m-1m	0.637	0.503	0.476	0.468
6m-1m	0.239	0.171	0.156	0.155
12m-1m	0.339	0.234	0.214	0.214
3y-1m	0.149	0.102	0.094	0.096
5y-1m	0.187	0.139	0.124	0.121
10y-1m	0.226	0.178	0.165	0.165

Fourth Break (1286–1769)

ℓ	0	4	8	12
3m-1m	0.568	0.506	0.402	0.341
6m-1m	0.138	0.138	0.126	0.119
12m-1m	0.134	0.125	0.113	0.112
3y-1m	0.556	0.448	0.379	0.356
5y-1m	0.176	0.162	0.148	0.148
10y-1m	0.156	0.144	0.128	0.127

Table 6.7: KPSS Test on M2 Splits: Yields

First Break (1 344)

ℓ	0	4	8	12
3-month	28.020	5.744	3.270	2.322
6-month	28.784	5.895	3.356	2.383
12-month	29.133	5.959	3.387	2.402
3-year	30.164	6.158	3.493	2.470
5-year	30.849	6.298	3.571	2.522
10-year	30.464	6.222	3.525	2.488

Second Break (345 423)

ℓ	0	4	8	12
3-month	2.277	0.503	0.313	0.249
6-month	2.042	0.448	0.279	0.221
12-month	2.048	0.460	0.294	0.239
3-year	4.762	1.046	0.646	0.501
5-year	5.625	1.227	0.751	0.572
10-year	6.243	1.355	0.830	0.635

Third Break (424 1285)

ℓ	0	4	8	12
3-month	28.505	5.780	3.263	2.297
6-month	29.171	5.906	3.331	2.343
12-month	31.868	6.447	3.634	2.554
3-year	40.750	8.223	4.619	3.234
5-year	43.956	8.861	4.968	3.473
10-year	49.027	9.868	5.520	3.849

Fourth Break (1286 1769)

ℓ	0	4	8	12
3-month	23.107	4.640	2.588	1.800
6-month	23.729	4.768	2.662	1.853
12-month	24.558	4.938	2.761	1.925
3-year	28.252	5.701	3.202	2.245
5-year	30.472	6.164	3.472	2.441
10-year	33.894	6.864	3.869	2.721

Table 6.8: KPSS Test on M2 Splits: Term Premium

First Break (1-344)

ℓ	0	4	8	12
12m-1m	2.051	0.451	0.268	0.198
3y-1m	1.673	0.367	0.220	0.163
5y-1m	4.218	0.896	0.526	0.385
10y-1m	9.895	2.066	1.200	0.870

Second Break (345-423)

ℓ	0	4	8	12
12m-1m	0.647	0.186	0.125	0.105
3y-1m	4.595	1.162	0.730	0.567
5y-1m	4.649	1.114	0.693	0.537
10y-1m	4.094	0.913	0.554	0.424

Third Break (424-1285)

ℓ	0	4	8	12
12m-1m	4.372	0.949	0.563	0.414
3y-1m	8.509	1.775	1.027	0.739
5y-1m	8.300	1.720	0.991	0.710
10y-1m	8.359	1.726	0.993	0.711

Fourth Break (1286-1769)

ℓ	0	4	8	12
12m-1m	4.256	0.885	0.509	0.364
3y-1m	3.334	0.685	0.392	0.280
5y-1m	4.748	0.966	0.546	0.386
10y-1m	5.732	1.159	0.651	0.456

Table 6.9: KPSS Test on Inflation Splits: Returns

First Break (1–253)

ℓ	0	4	8	12
1-month	15.426	4.162	2.443	1.748
3-month	13.132	4.052	2.407	1.723
6-month	5.438	2.437	1.638	1.256
12-month	2.590	1.268	0.962	0.790
3-year	0.324	0.169	0.139	0.120
5-year	0.243	0.149	0.122	0.107
10-year	0.088	0.057	0.067	0.070

Second Break (254–549)

ℓ	0	4	8	12
1-month	6.404	1.495	0.872	0.624
3-month	2.312	0.971	0.640	0.493
6-month	0.574	0.376	0.312	0.285
12-month	0.305	0.210	0.182	0.166
3-year	0.314	0.224	0.204	0.191
5-year	0.407	0.277	0.257	0.251
10-year	0.203	0.156	0.158	0.152

Third Break (550–1042)

ℓ	0	4	8	12
1-month	19.786	5.210	3.088	2.243
3-month	7.598	3.383	2.354	1.873
6-month	2.420	1.375	1.126	1.030
12-month	0.585	0.356	0.325	0.333
3-year	0.058	0.039	0.038	0.043
5-year	0.047	0.034	0.032	0.033
10-year	0.037	0.027	0.027	0.030

Fourth Break (1043–1769)

ℓ	0	4	8	12
1-month	29.155	7.338	4.252	3.022
3-month	20.395	7.141	4.376	3.239
6-month	10.000	5.126	3.392	2.614
12-month	4.271	3.029	2.240	1.842
3-year	2.629	1.779	1.354	1.148
5-year	1.802	1.432	1.159	1.019
10-year	1.017	0.906	0.758	0.685

Table 6.10: KPSS Test on Inflation Splits: Excess Returns

First Break (1 253)

ℓ	0	4	8	12
3m-1m	2.045	1.416	1.100	0.900
6m-1m	0.736	0.489	0.401	0.353
12m-1m	0.708	0.398	0.330	0.290
3y-1m	0.295	0.157	0.130	0.112
5y-1m	0.295	0.182	0.150	0.132
10y-1m	0.149	0.096	0.112	0.119

Second Break (254 549)

ℓ	0	4	8	12
3m-1m	0.266	0.232	0.208	0.203
6m-1m	0.169	0.144	0.143	0.156
12m-1m	0.111	0.084	0.078	0.075
3y-1m	0.378	0.273	0.250	0.235
5y-1m	0.470	0.321	0.299	0.292
10y-1m	0.239	0.184	0.186	0.179

Third Break (550 1042)

ℓ	0	4	8	12
3m-1m	0.268	0.229	0.239	0.252
6m-1m	0.051	0.037	0.035	0.037
12m-1m	0.083	0.055	0.053	0.057
3y-1m	0.147	0.100	0.098	0.110
5y-1m	0.106	0.078	0.073	0.075
10y-1m	0.133	0.098	0.096	0.106

Fourth Break (1043 1769)

ℓ	0	4	8	12
3m-1m	2.099	1.425	1.046	0.889
6m-1m	1.298	0.983	0.749	0.640
12m-1m	0.970	0.824	0.664	0.584
3y-1m	1.514	1.069	0.831	0.717
5y-1m	1.148	0.943	0.779	0.697
10y-1m	0.689	0.625	0.529	0.483

Table 6.11: KPSS Test on Inflation Splits: Yields

First Break (1–253)

ℓ	0	4	8	12
3-month	21.670	4.387	2.471	1.739
6-month	20.751	4.204	2.370	1.671
12-month	20.428	4.141	2.336	1.646
3-year	20.338	4.127	2.328	1.638
5-year	20.631	4.190	2.364	1.662
10-year	20.191	4.139	2.350	1.655

Second Break (254–549)

ℓ	0	4	8	12
3-month	7.667	1.568	0.893	0.634
6-month	7.299	1.497	0.855	0.610
12-month	6.279	1.294	0.743	0.533
3-year	4.472	0.925	0.533	0.384
5-year	5.296	1.096	0.632	0.453
10-year	7.990	1.653	0.951	0.681

Third Break (550–1042)

ℓ	0	4	8	12
3-month	29.948	6.109	3.466	2.452
6-month	30.747	6.264	3.553	2.513
12-month	32.215	6.564	3.723	2.632
3-year	35.371	7.199	4.069	2.866
5-year	36.121	7.348	4.147	2.916
10-year	38.147	7.748	4.363	3.059

Fourth Break (1043–1769)

ℓ	0	4	8	12
3-month	42.767	8.698	4.894	3.424
6-month	44.000	8.948	5.036	3.527
12-month	44.987	9.140	5.144	3.606
3-year	49.161	9.994	5.634	3.956
5-year	50.241	10.212	5.756	4.044
10-year	52.057	10.575	5.958	4.183

Table 6.12: KPSS Test on Inflation Splits: Term Premium

First Break (1-253)

ℓ	0	4	8	12
12m-1m	0.741	0.184	0.123	0.098
3y-1m	7.316	1.704	1.089	0.840
5y-1m	14.606	3.155	1.883	1.386
10y-1m	18.337	3.819	2.209	1.589

Second Break (254-549)

ℓ	0	4	8	12
12m-1m	5.968	1.338	0.789	0.577
3y-1m	17.013	3.577	2.051	1.454
5y-1m	18.126	3.746	2.131	1.503
10y-1m	17.982	3.679	2.085	1.469

Third Break (550-1042)

ℓ	0	4	8	12
12m-1m	10.529	2.296	1.358	0.993
3y-1m	3.933	0.834	0.489	0.355
5y-1m	5.030	1.057	0.616	0.447
10y-1m	5.893	1.230	0.715	0.517

Fourth Break (1043-1769)

ℓ	0	4	8	12
12m-1m	3.440	0.733	0.436	0.325
3y-1m	14.702	3.035	1.753	1.268
5y-1m	9.582	1.964	1.123	0.803
10y-1m	6.426	1.312	0.746	0.529

Table 6.13: KPSS Tests on Markov Switching Samples

3-month Returns

start	end	0	4	8	12
1	847	9.884	4.178	2.822	2.117
848	1198	0.702	0.391	0.303	0.271
1199	1769	11.265	4.307	2.570	1.838

10-year Returns

start	end	0	4	8	12
1	280	0.216	0.144	0.157	0.156
281	1769	0.223	0.182	0.166	0.164

3-year Excess Returns

start	end	0	4	8	12
1	924	0.327	0.214	0.201	0.198
925	1191	0.125	0.090	0.080	0.077
1092	1769	0.535	0.398	0.324	0.294

3-month Yields

start	end	0	4	8	12
1	852	33.254	6.774	3.832	2.700
853	1262	16.651	3.443	1.988	1.435
1263	1378	7.614	1.722	1.038	0.777
1379	1511	10.545	2.200	1.275	0.925
1512	1770	10.888	2.207	1.244	0.875

10-year Yields

start	end	0	4	8	12
1	855	74.474	15.026	8.418	5.874
856	1258	6.729	1.386	0.793	0.566
1259	1322	5.240	1.145	0.704	0.540
1322	1505	5.578	1.198	0.721	0.546
1506	1770	5.335	1.097	0.629	0.453

3-year Term Premium

start	end	0	4	8	12
1	469	8.444	1.851	1.100	0.807
470	567	3.432	0.748	0.453	0.340
568	679	7.711	1.719	1.048	0.777
680	866	11.102	2.396	1.409	1.036
867	1025	3.596	0.787	0.485	0.376
1026	1396	7.418	1.576	0.940	0.706
1397	1513	8.069	1.753	1.055	0.793
1514	1733	2.488	0.525	0.309	0.229
1734	1770	2.337	0.539	0.353	0.300

Table 6.14: Cramer-von Misés Statistics

Inflation Sample Splits

ℓ	0	4	8	12	brks+1	5% cv*
infl	1.115	0.329	0.203	0.148	4	1.237
ret3m	29.368	10.756	6.828	5.105	4	1.237
ret10y	1.514	1.225	1.123	1.111	4	1.237
xret3y	1.931	1.338	1.194	1.176	4	1.237
yld3m	70.439	14.218	7.981	5.586	4	1.237
yld10y	70.590	14.182	7.916	5.508	4	1.237
tp3y	22.580	4.680	2.690	1.923	4	1.237

* cv's taken from Busetti and Harvey (2001), Nyblom (1989)

Markov-Switching Sample Splits

ℓ	0	4	8	12	brks+1	5% cv*
ret3m	8.905	3.261	2.070	1.548	3	1.000
ret10y	0.289	0.234	0.214	0.212	2	0.748
xret3y	0.963	0.667	0.596	0.587	3	1.000
yld3m	29.385	5.931	3.330	2.330	5	< 1.686 (6)
yld10y	29.469	5.920	3.305	2.299	5	< 1.686 (6)
tp3y	29.196	6.051	3.477	2.486	9	2.116 < 2.533 (10)

* cv's taken from Busetti and Harvey (2001), Nyblom (1989)

Figure 6.1: Autocorrelation Functions for I(*d*) and AR(1) Processes

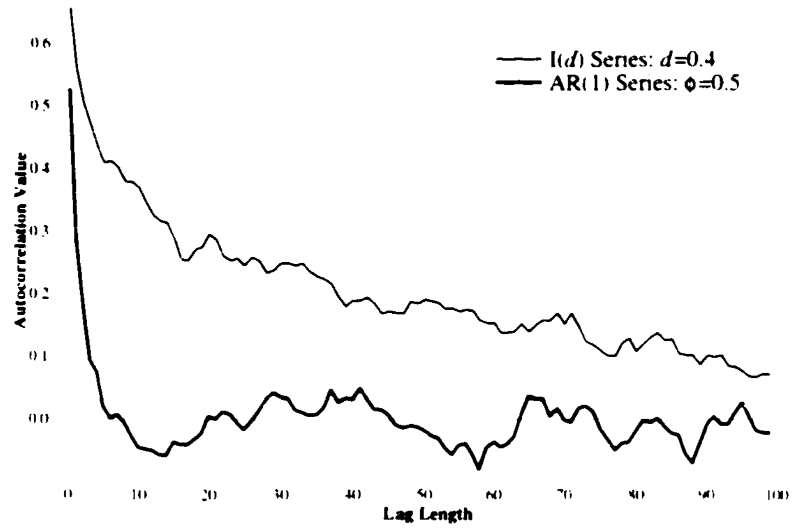


Figure 6.2: Partial Sums for I(*d*) and AR(1) Processes

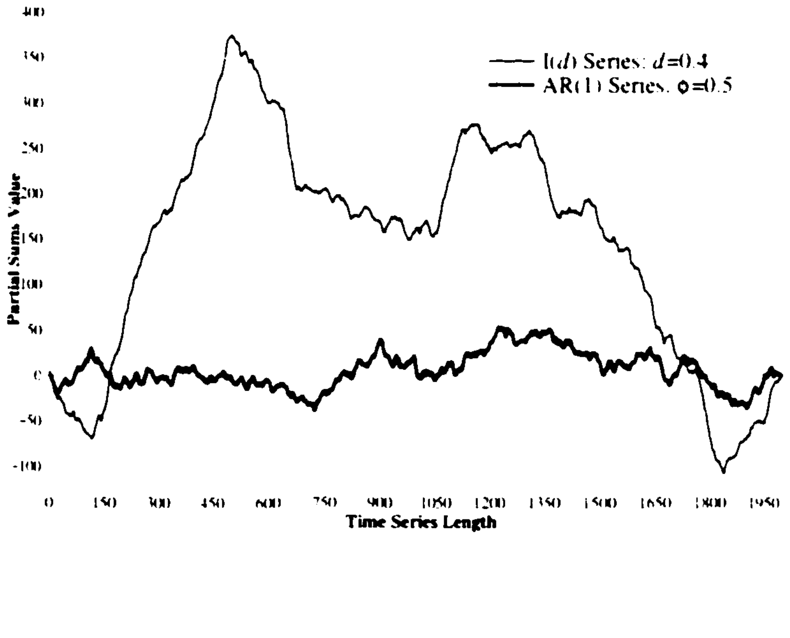


Figure 6.3: Inflation with Estimated Break Points

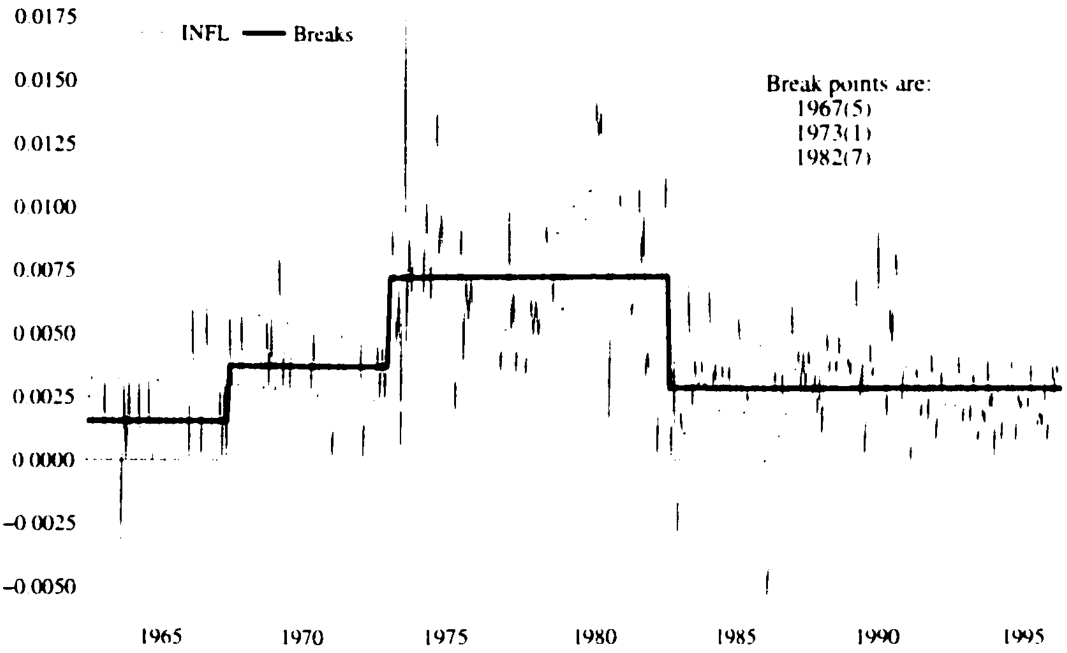


Figure 6.4: 3-Month Bond Return, State Probability

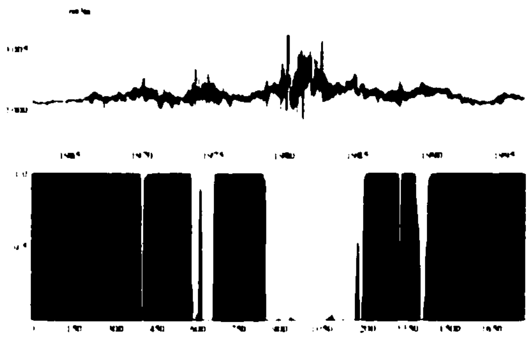


Figure 6.5: 10-Year Return, State Probability

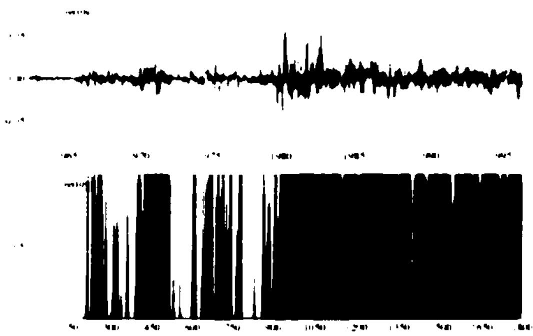


Figure 6.6: 3 Year Bond Excess Return, State Probability

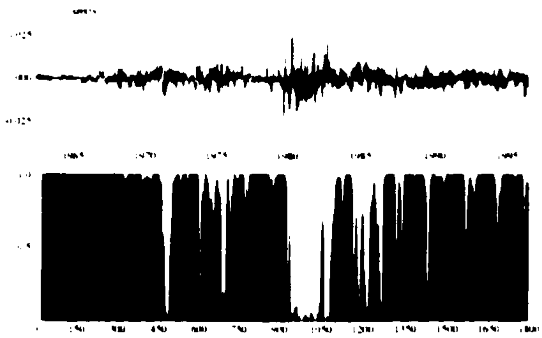


Figure 6.7: 3-Month Bill Yield, State Probability

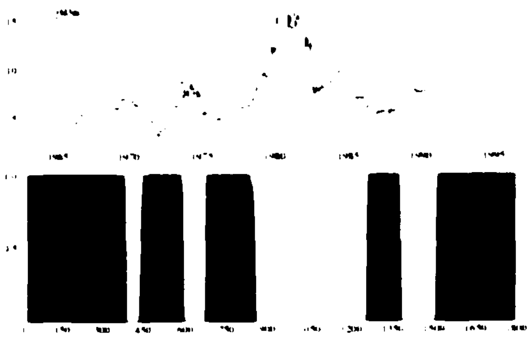


Figure 6.8: 10-Year Bond Yield, State Probability

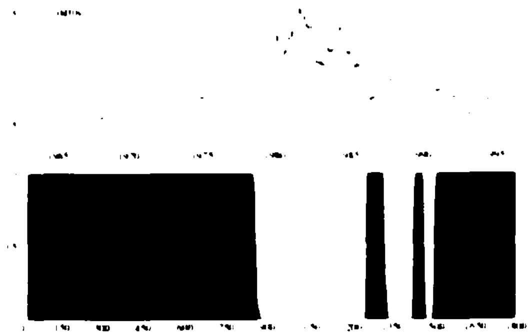
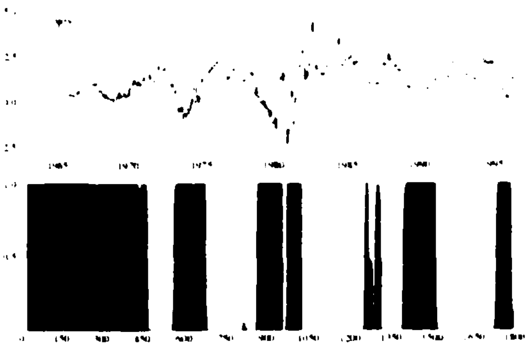


Figure 6.9: 3-Yr Bond Term Premium, State Probability



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